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Permutation transformations of tensors with an application

Yao-Tang Li^{*†}, Zheng-Bo Li[†], Qi-Long Liu[†] and Qiong Liu[†]

*Correspondence:

liyaotang@ynu.edu.cn

[†]Yao-Tang Li, Zheng-Bo Li, Qi-Long Liu and Qiong Liu contributed equally to this work

School of Mathematics and Statistics, Yunnan University, Kunming 650091, People's Republic of China

Abstract

The permutation transformation of tensors is introduced and its basic properties are discussed. The invariance under permutation transformations is studied for some important structure tensors such as symmetric tensors, positive definite (positive semidefinite) tensors, Z -tensors, M -tensors, Hankel tensors, P -tensors, B -tensors and H -tensors. Finally, as an application of permutation transformations of tensors, the canonical form theorem of tensors is given. The theorem shows that some problems of higher dimension tensors can be translated into the corresponding problems of lower dimension weakly irreducible tensors so as to handle easily.

Keywords: Permutation transformation, Structure tensor, Weakly irreducible tensor, Canonical form

Mathematics Subject Classification: 15A69, 12E05, 12E10

Background

The study of tensors with their various applications has attracted extensive attention and interest, since the work of Qi (2005) and Lim (2005). Lately, the research topic on structure tensors has also attracted much attention, such as symmetric tensors (Qi 2005), $P(P_0)$ -tensors (Song and Qi 2014), $B(B_0)$ -tensors (Song and Qi 2014), Z -tensors (Zhang et al. 2014), (strong) M -tensors (Zhang et al. 2014), H -tensors (Li et al. 2014) and so on. In the researches on tensors with its application, the reducibility and higher dimension of tensors are two important factors to cause difficulties. Therefore, it is interesting that how to translate problems of higher dimension reducible tensors into the corresponding problems of lower dimension irreducible tensors.

As we all know, the permutation transformation of matrices plays a very important role in linear algebra and matrix theory. Some problems of higher dimension reducible matrices can be translated into the corresponding problems of lower dimension irreducible matrices by using the permutation transformation of matrices. Inspired by this, we introduce permutation transformations of tensors, and discuss its basic properties and and their applications in this paper.

In the next section, we will introduce the permutation transformation of tensors and give its expression. In third section, we will discuss basic properties of permutation transformations of tensors. In fourth section, we will discuss the invariance under

permutation transformations for some important structure tensors such as symmetric tensors, positive definite tensors, \mathcal{M} -tensors, Hankel tensors, P -tensors, B -tensors, H -tensors and so on. In fifth section, we will give the canonical form theorem of tensors and a numerical example which shows that some problems of higher dimension tensors can be translated into the corresponding problems of lower dimension weakly irreducible tensors by using permutation transformations. Finally, we draw some conclusions in the last section.

Permutation transformations of tensors and its expression

For a positive integer n , let $[n] = \{1, 2, \dots, n\}$. An order m tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{n_1 \times n_2 \times \dots \times n_m}$ is a multidimensional array with $n_1 n_2 \dots n_m$ entries, where $i_j \in [n_j]$, $j \in [m]$. Especially, an order m dimension n tensor $\mathcal{A} = (a_{i_1 \dots i_m})$ over the complex field \mathbb{C} (real field \mathbb{R}) consists of n^m complex (real) entries:

$$a_{i_1 \dots i_m} \in \mathbb{C} \ (\mathbb{R}),$$

where $i_j \in [n]$ for $j \in [m]$ (Chang et al. 2008; De Lathauwer et al. 2000; Liu et al. 2010; Ng et al. 2009; Zhang and Golub 2001). It is obvious that a matrix is an order 2 tensor. We shall denote the set of all complex (real) order m dimension n tensors by $\mathbb{C}^{[m,n]}$ ($\mathbb{R}^{[m,n]}$, respectively).

Definition 1 Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$, $\mathcal{B} = (b_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$, and $k \in \mathbb{C}$. Define

- (i) $\mathcal{A} + \mathcal{B} = (a_{i_1 \dots i_m} + b_{i_1 \dots i_m})$.
- (ii) $k\mathcal{A} = (ka_{i_1 \dots i_m})$.

Remark 1 Obviously, both $\mathbb{C}^{[m,n]}$ and $\mathbb{R}^{[m,n]}$ are linear spaces about the addition and the multiplication in Definition 1.

Definition 2 (Qi 2005) A tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ is called a symmetric tensor if its entries $a_{i_1 \dots i_m}$ are invariant under any permutation of their indices.

Denote the set of all real order m dimension n symmetric tensors by $\mathbb{S}^{[m,n]}$. Furthermore, $\mathbb{S}^{[m,n]}$ is a linear subspace of $\mathbb{R}^{[m,n]}$. An order m dimension n tensor is called the unit tensor (Yang and Yang 2010), denoted by \mathcal{I} , if its entries are $\delta_{i_1 \dots i_m}$ for $i_1, \dots, i_m \in [n]$, where

$$\delta_{i_1 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ and $x \in \mathbb{R}^n$. Then $\mathcal{A}x^m$ is a homogeneous polynomial of degree m , defined by

$$\mathcal{A}x^m = \sum_{i_1, \dots, i_m \in [n]} a_{i_1 \dots i_m} x_{i_1} \dots x_{i_m}.$$

A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called positive semidefinite (Song and Qi 2014) if for any vector $x \in \mathbb{R}^n$, $\mathcal{A}x^m \geq 0$, and it is called positive definite if for any nonzero vector $x \in \mathbb{R}^n$, $\mathcal{A}x^m > 0$.

Now, we give the definition of permutation transformation of tensors.

Definition 3 Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$ and π be a permutation on $[n]$, we define $P_\pi : \mathbb{C}^{[m,n]} \rightarrow \mathbb{C}^{[m,n]}$ by

$$P_\pi(\mathcal{A}) = (a_{\pi(i_1) \dots \pi(i_m)}).$$

P_π is called as a permutation transformation on $\mathbb{C}^{[m,n]}$, and is simply called as a permutation transformation. $P_\pi(\mathcal{A})$ is called as the image of \mathcal{A} under P_π .

Remark 2 P_π is called as a permutation transformation on $\mathbb{R}^{[m,n]}$ if $\mathbb{C}^{[m,n]}$ is replaced by $\mathbb{R}^{[m,n]}$ in Definition 3.

Definition 4 Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$ and π^{-1} be the inverse permutation of π on $[n]$, we define $P_\pi^{-1} : \mathbb{C}^{[m,n]} \rightarrow \mathbb{C}^{[m,n]}$ by

$$P_\pi^{-1}(\mathcal{A}) = P_{\pi^{-1}}(\mathcal{A}).$$

P_π^{-1} is called as the inverse permutation transformation of P_π on $\mathbb{C}^{[m,n]}$, and is simply called as the inverse permutation transformation.

For further discussing property of the permutation transformation of tensors, we introduce the following general product of two n -dimensional tensors defined in Shao (2013). For the sake of simplicity, we sometime use the following “condensed notation” for the subscripts of the tensor. For example, we will write $a_{i_1 i_2 \dots i_m}$ as $a_{i_1 \alpha}$, where $\alpha = i_2 \dots i_m \in [n]^{m-1}$ and $[n]^{m-1}$ is $m - 1$ dimensional array whose every element varies from 1 to n .

Definition 5 (Shao 2013) Let \mathcal{A} and \mathcal{B} be order $m \geq 2$ and order $k \geq 1$, dimension n tensors, respectively. Define the product $\mathcal{A} \cdot \mathcal{B}$ (sometimes simplified as \mathcal{AB}) to be the following tensor \mathcal{C} of order $(m - 1)(k - 1) + 1$ and dimension n ,

$$c_{i \alpha_1 \dots \alpha_{m-1}} = \sum_{i_2, \dots, i_m \in [n]} a_{i i_2 \dots i_m} b_{i_2 \alpha_1} \dots b_{i_m \alpha_{m-1}},$$

where $i \in [n], \alpha_1, \dots, \alpha_{m-1} \in [n]^{k-1}$.

Especially, when $P = (p_{ij})$ and $Q = (q_{ij})$ are both matrices, we have the following formula (Shao 2013):

$$(PAQ)_{i_1 \dots i_m} = \sum_{j_1, \dots, j_m \in [n]} a_{j_1 \dots j_m} p_{i_1 j_1} q_{j_2 i_2} \dots q_{j_m i_m}.$$

Definition 6 (Shao 2013) Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{[m,n]}$. If there exists a permutation π on the set $[n]$, and the corresponding permutation matrix $P = (P_{ij})$ (where $P_{ij} = 1 \iff j = \pi(i); P_{ij} = 0$, otherwise.) such that $\mathcal{B} = P\mathcal{A}P^T$, then we say that \mathcal{A} and \mathcal{B} are permutational similar.

Remark 3 (Shao 2013) If \mathcal{A}, \mathcal{B} are permutational similar, then

- (i) $b_{i_1 \dots i_m} = a_{\pi(i_1) \dots \pi(i_m)}$,
- (ii) $P\mathcal{I}P^T = \mathcal{I}$.

Definition 7 (Shao 2013) Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{[m,n]}$. Suppose that there exist two matrices P and Q of dimension n with $P\mathcal{I}Q = \mathcal{I}$ such that $\mathcal{B} = P\mathcal{A}Q$, then we say that the two tensors \mathcal{A} and \mathcal{B} are similar.

Now, we present the relationship between the permutation transformation of tensors and permutational similar.

Theorem 1 Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{[m,n]}$. Then \mathcal{A} and \mathcal{B} are permutational similar if and only if there exists a permutation π on $[n]$ such that $\mathcal{B} = P_\pi(\mathcal{A})$.

Proof Assume that \mathcal{A} and \mathcal{B} are permutational similar. By Remark 3, there exists a permutation π on $[n]$ such that $b_{i_1 \dots i_m} = a_{\pi(i_1) \dots \pi(i_m)}$. Define a permutation transformation $P_\pi : \mathbb{C}^{[m,n]} \rightarrow \mathbb{C}^{[m,n]}$ by

$$P_\pi(\mathcal{A}) = (a_{\pi(i_1) \dots \pi(i_m)}),$$

which implies $\mathcal{B} = P_\pi(\mathcal{A})$.

On the other hand, assume that there exists a permutation transformation P_π such that $\mathcal{B} = P_\pi(\mathcal{A})$, then

$$b_{i_1 \dots i_m} = a_{\pi(i_1) \dots \pi(i_m)}.$$

Let permutation matrix $P = P_\pi$ ($P_{ij} = 1 \iff j = \pi(i)$; $P_{ij} = 0$, otherwise.) corresponding to π . Then

$$\begin{aligned} (P\mathcal{A}P^T)_{i_1 \dots i_m} &= \sum_{j_1, \dots, j_m \in [n]} a_{j_1 \dots j_m} P_{i_1 j_1} (P^T)_{j_2 i_2} \dots (P^T)_{j_m i_m} \\ &= a_{\pi(i_1) \dots \pi(i_m)}. \end{aligned}$$

Hence, $\mathcal{B} = P\mathcal{A}P^T$. Thus, \mathcal{A} and \mathcal{B} are permutational similar. □

By Theorem 1 and Remark 3, we have the following expression theorem of permutational transformation of tensors.

Theorem 2 Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$, and π be a permutation on $[n]$. Then

$$P_\pi(\mathcal{A}) = P\mathcal{A}P^T,$$

where $P_{ij} = 1 \iff j = \pi(i)$; $P_{ij} = 0$, otherwise.

Basic properties of permutation transformations

Now, we discuss basic properties of permutation transformation of tensors. Firstly, we present some definitions and a lemma, which are needed in the subsequent analysis. For an n -dimensional vector $x = (x_1, x_2, \dots, x_n)$, real or complex, we define the n -dimensional vector:

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m} \right)_{1 \leq i \leq n},$$

and the n -dimensional vector:

$$x^{[m-1]} := \left(x_i^{m-1} \right)_{1 \leq i \leq n}.$$

The following two definitions were first introduced and studied by Lim (2005) and Qi (2005).

Definition 8 (Lim 2005; Qi 2005) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of \mathcal{A} if they satisfy the equation

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{1}$$

We call (λ, x) an H-eigenpair if they are both real.

The set of all eigenvalues of \mathcal{A} is denoted by $\sigma(\mathcal{A})$ and it is called the spectral of \mathcal{A} . Let $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$. It is called the spectral radius of \mathcal{A} .

Definition 9 (Lim 2005; Qi 2005) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenvalue and E-eigenvector (or simply E-eigenpair) of \mathcal{A} if they satisfy the equation

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^T x = 1. \end{cases} \tag{2}$$

We call (λ, x) an Z-eigenpair if they are both real.

Definition 10 (Zhang et al. 2014) A tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ is called strictly diagonally dominant if

$$|a_{ii \dots i}| > \sum_{\substack{i_2, \dots, i_m \in [n], \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|, \quad \forall i \in [n].$$

Definition 11 (Bu et al. 2014) A tensor $\mathcal{A} \in \mathbb{C}^{n_1 \times \dots \times n_k}$ is said to have rank-one if there exist nonzero $a_i \in \mathbb{C}^{n_i} (i = 1, \dots, k)$ such that $\mathcal{A} = a_1 \otimes a_2 \otimes \dots \otimes a_k$, where $a_1 \otimes a_2 \otimes \dots \otimes a_k$ is the segre outer product of $a_1 \in \mathbb{C}^{n_1}, \dots, a_k \in \mathbb{C}^{n_k}$ with entries $a_{i_1 \dots i_k} = (a_1)_{i_1} \dots (a_k)_{i_k}$. The rank of a tensor \mathcal{A} , denoted by $rank(\mathcal{A})$, is defined to be the smallest r such that \mathcal{A} can be written as a sum of r rank-one tensors. Especially, if $\mathcal{A} = 0$, then $rank(\mathcal{A}) = 0$.

Lemma 1 (Qi 2005) Assume that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is an even-order symmetric tensor. The following conclusions hold for \mathcal{A} ,

- (i) \mathcal{A} always has H-eigenvalues. \mathcal{A} is positive definite (positive semidefinite) if and only if all of its H-eigenvalues are positive (nonnegative).
- (ii) \mathcal{A} always has Z-eigenvalues. \mathcal{A} is positive definite (positive semidefinite) if and only if all of its Z-eigenvalues are positive (nonnegative).

Next, we present some basic properties of permutation transformation of tensor as follows.

Theorem 3 Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{[m,n]} (\mathbb{R}^{[m,n]})$, and π be a permutation on $[n]$. Then

- (i) $P_\pi(\mathcal{A} + \mathcal{B}) = P_\pi(\mathcal{A}) + P_\pi(\mathcal{B})$.
- (ii) $P_\pi(k\mathcal{A}) = kP_\pi(\mathcal{A})$, where $k \in \mathbb{C} (\mathbb{R})$.
- (iii) $P_\pi(\mathcal{I}) = \mathcal{I}$.
- (iv) $P_\pi^{-1}(P_\pi(\mathcal{A})) = \mathcal{A}$.
- (v) $P_\pi(P_\pi^{-1}(\mathcal{A})) = \mathcal{A}$.
- (vi) $\sigma(P_\pi(\mathcal{A})) = \sigma(\mathcal{A})$.
- (vii) $\rho(P_\pi(\mathcal{A})) = \rho(\mathcal{A})$.
- (viii) $rank(P_\pi(\mathcal{A})) = rank(\mathcal{A})$.
- (ix) If $\mathcal{A} \in \mathbb{S}^{[m,n]}$, then $P_\pi(\mathcal{A}) \in \mathbb{S}^{[m,n]}$.
- (x) If \mathcal{A} is a strictly diagonally dominant tensor, then $P_\pi(\mathcal{A})$ is also a strictly diagonally dominant tensor.
- (xi) If m is even and $\mathcal{A} \in \mathbb{S}^{[m,n]}$ is positive definite (positive semidefinite) tensor, then $P_\pi(\mathcal{A}) \in \mathbb{S}^{[m,n]}$ and is also a positive definite (positive semidefinite) tensor.

Proof (i) $P_\pi(\mathcal{A} + \mathcal{B})_{i_1 \dots i_m} = a_{\pi(i_1) \dots \pi(i_m)} + b_{\pi(i_1) \dots \pi(i_m)} = (P_\pi(\mathcal{A}) + P_\pi(\mathcal{B}))_{i_1 \dots i_m}$.

(ii) $P_\pi(k\mathcal{A})_{i_1 \dots i_m} = ka_{\pi(i_1) \dots \pi(i_m)} = kP_\pi(\mathcal{A})_{i_1 \dots i_m}$.

(iii) Since $P_\pi(\mathcal{I})_{i_1 \dots i_m} = \mathcal{I}_{\pi(i_1) \dots \pi(i_m)}$ and $\delta_{\pi(i_1) \dots \pi(i_m)} = 1$ if and only if $\delta_{i_1 \dots i_m} = 1$, then $P_\pi(\mathcal{I}) = \mathcal{I}$.

From the Definition 3, it is easy to obtain (iv) and (v) are hold.

(vi) By Theorem 1, $P_\pi(\mathcal{A})$ and \mathcal{A} are permutational similar. By Theorem 2.1 in Shao (2013),

$$\phi_{P_\pi(\mathcal{A})}(\lambda) = \phi_{\mathcal{A}}(\lambda),$$

where $\phi_{\mathcal{A}}(\lambda)$ is the characteristic polynomial of the tensor \mathcal{A} . In Qi (2005), Qi has proved that a number $\lambda \in \mathbb{C}$ is an eigenvalue of \mathcal{A} if and only if it is a root of $\phi_{\mathcal{A}}(\lambda)$. Hence, similar tensors have the same eigenvalues. Then

$$\sigma(\mathcal{A}) = \sigma(P_\pi(\mathcal{A})).$$

(vii) It is easy to be got from the results of (vi).

(viii) Let $rank(\mathcal{A}) = r$, and $P_\pi(\mathcal{A}) = (a_{\pi(i_1) \dots \pi(i_m)})$, where π is a permutation on $[n]$.

Case 1. If $r = 1$, then there exists $a_i \in \mathbb{C}^{n_i}$ ($i \in [m], a_i \neq 0$) such that

$$\mathcal{A} = a_1 \otimes a_2 \otimes \dots \otimes a_m$$

which implies,

$$a_{i_1 \dots i_m} = (a_1)_{i_1} (a_2)_{i_2} \dots (a_m)_{i_m}.$$

Therefore,

$$\begin{aligned} a_{\pi(i_1) \pi(i_2) \dots \pi(i_m)} &= (a_1)_{\pi(i_1)} (a_2)_{\pi(i_2)} \dots (a_m)_{\pi(i_m)} \\ &= (P_\pi(a_1))_{i_1} (P_\pi(a_2))_{i_2} \dots (P_\pi(a_m))_{i_m}. \end{aligned}$$

Hence, $P_\pi(\mathcal{A}) = P_\pi(a_1) \otimes P_\pi(a_2) \otimes \dots \otimes P_\pi(a_m)$. Since $P_\pi(a_i) \in \mathbb{C}^{n_i}$ and $P_\pi(a_i) \neq 0$, then $rank(P_\pi(\mathcal{A})) = 1$.

Case 2. If $r > 1$, then \mathcal{A} can be written at least as a sum of r rank-one tensors. Let $\mathcal{A} = \sum_{i \in [r]} \mathcal{A}_i$, where $\mathcal{A}_i \in \mathbb{C}^{[m,n]}$, $rank(\mathcal{A}_i) = 1, i \in [r]$. Then,

$$P_\pi(\mathcal{A}) = \sum_{i \in [r]} P_\pi(\mathcal{A}_i).$$

By the results of Case 1, we have

$$rank(P_\pi(\mathcal{A}_i)) = 1, i \in [r].$$

Hence, $rank(P_\pi(\mathcal{A})) \leq r$. Next, we will prove that $rank(P_\pi(\mathcal{A})) < r$ is impossible. Suppose that $rank(P_\pi(\mathcal{A})) = r' < r$. Then $P_\pi(\mathcal{A})$ can be written at least as a sum of r' rank-one tensors as follows

$$P_\pi(\mathcal{A}) = \sum_{i \in [r']} \mathcal{D}_i, \text{ where } \mathcal{D}_i \in \mathbb{C}^{[m,n]}, \text{ } rank(\mathcal{D}_i) = 1, i \in [r'],$$

and π^{-1} be the inverse transformation of π on $[n]$. Then from (iv),

$$\mathcal{A} = P_\pi^{-1}(P_\pi(\mathcal{A})) = \sum_{i \in [r']} P_{\pi^{-1}}(\mathcal{D}_i).$$

From case 1, $rank(P_\pi^{-1}(\mathcal{D}_i)) = 1$, so $rank(\mathcal{A}) \leq r' < r$. It's a contradiction. Therefore, $rank(P_\pi(\mathcal{A})) = r$.

- (ix) It is easy to be proved from the definition of symmetric tensors.
- (x) Suppose that \mathcal{A} is a strictly diagonally dominant tensor, then

$$\begin{aligned} |P_\pi(\mathcal{A})_{ii\dots i}| &= |\mathcal{A}_{\pi(i)\pi(i)\dots\pi(i)}| \\ &> \sum_{\substack{i_2, \dots, i_m \in [n], \\ \delta_{\pi(i)i_2\dots i_m} = 0}} |a_{\pi(i)i_2\dots i_m}| \\ &= \sum_{\substack{i_2, \dots, i_m \in [n], \\ \delta_{ii_2\dots i_m} = 0}} |P_\pi(\mathcal{A})_{ii_2\dots i_m}|, \forall i \in [n]. \end{aligned}$$

Thus, $P_\pi(\mathcal{A})$ is also a strictly diagonally dominant tensor.

- (xi) Suppose that \mathcal{A} is positive definite (semidefinite), then all H-eigenvalues of \mathcal{A} are positive (nonnegative). By (vi) and (ix) of Theorem 3, $P_\pi(\mathcal{A})$ is an even-order symmetric tensor, and $\sigma(P_\pi(\mathcal{A})) = \sigma(\mathcal{A})$. From the results above, we know that all H-eigenvalues of $P_\pi(\mathcal{A})$ are positive (nonnegative). Then, by Lemma 1, $P_\pi(\mathcal{A})$ is positive definite (positive semidefinite). □

Remark 4 By the results of (i) and (ii) of Theorem 3, we know that a permutation transformation of tensor is a linear transformation on $\mathbb{C}^{[m,n]}$ ($\mathbb{R}^{[m,n]}$).

Permutation transformation on some structure tensors

It is universally acknowledged that some structure tensors with good properties have been well studied, such as nonnegative tensor, symmetric tensor, positive definite (positive semidefinite) tensor, Z -tensor, (strong) M -tensor, Hankel tensor, $P(P_0)$ -tensor, $B(B_0)$ -tensor and H -tensor. In this section, we discuss the invariance under permutation transformations for some important structure tensors.

Definition 12 (Chang et al. 2008) A tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ is called a nonnegative tensor, denoted by $\mathcal{A} \geq 0$, if each entry is nonnegative.

Definition 13 (Zhang et al. 2014) We call a tensor \mathcal{A} as an Z -tensor, if all of its off-diagonal entries are non-positive, which is equivalent to write $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $s > 0$ and \mathcal{B} is a nonnegative tensor.

From Definition 12, we easily get the following two lemmas.

Lemma 2 Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonnegative tensor, π be a permutation on $[n]$. Then $P_\pi(\mathcal{A})$ is also a nonnegative tensor.

Lemma 3 Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be an Z -tensor, and π be a permutation on $[n]$. Then $P_\pi(\mathcal{A})$ is also an Z -tensor.

Definition 14 (Zhang et al. 2014) We call an Z -tensor $\mathcal{A} = s\mathcal{I} - \mathcal{B} (\mathcal{B} \geq 0)$ as an M -tensor if $s \geq \rho(\mathcal{B})$; we call it as a strong M -tensor if $s > \rho(\mathcal{B})$, where $\rho(\mathcal{B})$ is the spectral radius of \mathcal{B} .

Theorem 4 Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be an (strong) M -tensor, and π be a permutation on $[n]$. Then $P_\pi(\mathcal{A})$ is also an (strong) M -tensor.

Proof Suppose that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is an (strong) M -tensor, then there exist a nonnegative tensor \mathcal{B} and a real number $s \geq \rho(\mathcal{B}) (s > \rho(\mathcal{B}))$ such that

$$\mathcal{A} = s\mathcal{I} - \mathcal{B}.$$

From Remark 3,

$$P_\pi(\mathcal{A}) = s\mathcal{I} - P_\pi(\mathcal{B}).$$

By Lemma 3, $P_\pi(\mathcal{A})$ is an Z -tensor. It follows from (vii) of Theorem 3 that $\rho(P_\pi(\mathcal{B})) = \rho(\mathcal{B})$, which implies

$$s \geq \rho(P_\pi(\mathcal{B})) \quad (s > \rho(P_\pi(\mathcal{B}))).$$

Therefore, $P_\pi(\mathcal{A})$ is an (strong) M -tensor. □

Definition 15 (Song and Qi 2014) Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$. We say that \mathcal{A} is

- (i) a P_0 -tensor if for any nonzero vector $x \in \mathbb{R}^n$, there exists $i \in [n]$ such that $x_i \neq 0$ and $x_i(\mathcal{A}x^{m-1})_i \geq 0$;
- (ii) a P -tensor if for any nonzero vector $x \in \mathbb{R}^n$, $\max_{i \in [n]} x_i(\mathcal{A}x^{m-1})_i > 0$.

Lemma 4 (Song and Qi 2014) A symmetric tensor is a $P(P_0)$ -tensor if and only if it is positive (semi)definite.

Theorem 5 Let $\mathcal{A} \in \mathbb{S}^{[m,n]}$ be an even-order $P(P_0)$ -tensor, and π be a permutation on $[n]$. Then $P_\pi(\mathcal{A})$ is also an even-order symmetric $P(P_0)$ -tensor.

Proof Suppose that \mathcal{A} is an even-order symmetric $P(P_0)$ -tensor. According to Lemma 4, \mathcal{A} is positive (semi)definite. It follows from (ix) of Theorem 3 that $P_\pi(\mathcal{A})$ is positive (semi)definite. Hence, by Lemma 4, $P_\pi(\mathcal{A})$ is a $P(P_0)$ -tensor. \square

Definition 16 (Song and Qi 2014) Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m, n]}$. We say that \mathcal{A} is a B -tensor, if for all $i \in [n]$,

$$\sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} > 0,$$

and

$$\frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} \right) > a_{ij_2 \dots j_m},$$

for all $(j_2, \dots, j_m) \neq (i, \dots, i)$. We say that \mathcal{A} is a B_0 -tensor, if for all $i \in [n]$,

$$\sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} \geq 0,$$

and

$$\frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} \right) \geq a_{ij_2 \dots j_m},$$

for all $(j_2, \dots, j_m) \neq (i, \dots, i)$.

Theorem 6 Let $\mathcal{A} \in \mathbb{R}^{[m, n]}$ be a $B(B_0)$ -tensor, and π be a permutation on $[n]$. Then $P_\pi(\mathcal{A})$ is also a $B(B_0)$ -tensor.

Proof Suppose that $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ is a B -tensor, then for all $i \in [n]$,

$$\sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} > 0,$$

and

$$\frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} \right) > a_{ij_2 \dots j_m},$$

for all $(j_2, \dots, j_m) \neq (i, \dots, i)$.

Let $P_\pi(\mathcal{A}) = (b_{i_1 i_2 \dots i_m})$. Then

$$\begin{aligned} \sum_{i_2, \dots, i_m \in [n]} b_{ii_2 \dots i_m} &= \sum_{i_2, \dots, i_m \in [n]} a_{\pi(i)\pi(i_2)\dots\pi(i_m)} \\ &= \sum_{\pi(i_2), \dots, \pi(i_m) \in [n]} a_{\pi(i)\pi(i_2)\dots\pi(i_m)} > 0, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m \in [n]} b_{ii_2 \dots i_m} \right) &= \frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m \in [n]} a_{\pi(i)\pi(i_2)\dots\pi(i_m)} \right) \\ &= \frac{1}{n^{m-1}} \left(\sum_{\pi(i_2), \dots, \pi(i_m) \in [n]} a_{\pi(i)\pi(i_2)\dots\pi(i_m)} \right) \\ &> a_{\pi(i)\pi(j_2)\dots\pi(j_m)} = b_{ij_2 \dots j_m}. \end{aligned}$$

Hence, $P_\pi(\mathcal{A})$ is a B -tensor. Similarly, we can prove that the case of B_0 -tensor. □

Definition 17 (Li et al. 2014) A tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{C}^{[m, n]}$ is called an H -tensor if there is an entrywise positive vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ such that for all $i \in [n]$,

$$|a_{i \dots i}| x_i^{m-1} > \sum_{\substack{i_2, \dots, i_m \in [n], \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| x_{i_2} \dots x_{i_m}. \tag{3}$$

Theorem 7 Let $\mathcal{A} \in \mathbb{C}^{[m, n]}$ be an H -tensor, and π be a permutation on $[n]$. Then $P_\pi(\mathcal{A})$ is also an H -tensor.

Proof Suppose that $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ is an H -tensor. Then there is an entrywise positive vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ such that for all $i \in [n]$, the Inequality 3 holds. Let $P_\pi(\mathcal{A}) = (b_{i_1 i_2 \dots i_m})$, where $b_{i_1 i_2 \dots i_m} = a_{\pi(i_1)\pi(i_2)\dots\pi(i_m)}$. Hence, for all $i \in [n]$,

$$\begin{aligned} |b_{i \dots i}| x_i^{m-1} &= |a_{\pi(i)\dots\pi(i)}| x_i^{m-1} \\ &> \sum_{\substack{i_2, \dots, i_m \in [n], \\ \delta_{\pi(i)i_2 \dots i_m} = 0}} |a_{\pi(i)i_2 \dots i_m}| x_{i_2} \dots x_{i_m} \\ &= \sum_{\substack{i_2, \dots, i_m \in [n], \\ \delta_{ii_2 \dots i_m} = 0}} |b_{ii_2 \dots i_m}| x_{i_2} \dots x_{i_m} \\ &= \sum_{\substack{i_2, \dots, i_m \in [n], \\ \delta_{ii_2 \dots i_m} = 0}} |b_{ii_2 \dots i_m}| x_{i_2} \dots x_{i_m}. \end{aligned}$$

Thus, $P_\pi(\mathcal{A})$ is an H -tensor. □

Canonical form of tensors

In this section, as an application of the permutation transformation of tensors, we would introduce some results which show that the canonical form theorem for matrices could be generalized to tensors. For the convenience discussion, we starts with the following definitions and lemmas.

Let $\mathcal{A} \in \mathbb{R}^{[m, n]}$ be a nonnegative tensor. The directed graph $G(\mathcal{A})$ (Chang et al. 2013; Friedland et al. 2013) associated to \mathcal{A} is the directed graph with vertices $1, 2, \dots, n$ and an edge from i to j if and only if $a_{ii_2 \dots i_m} \neq 0$ for some $i_l = j, l = 2, 3, \dots, m$.

Definition 18 (Chang et al. 2013; Friedland et al. 2013; Hu et al. 2014) A nonnegative tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called weakly irreducible if the associate directed graph $G(\mathcal{A})$ is strongly connected. A tensor \mathcal{A} is said to be weakly irreducible if $|\mathcal{A}|$ is weakly irreducible, where $|\mathcal{A}|$ denote the tensor whose (i_1, \dots, i_m) -th entry is defined as $|a_{i_1 \dots i_m}|$.

Definition 19 (Shao et al. 2013) Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$. If there exists some integer k with $1 \leq k \leq n - 1$ such that

$$a_{i_1 i_2 \dots i_m} = 0,$$

for all $i_1 \leq k$ and at least one of $\{i_2, \dots, i_m\}$ is greater than k , then \mathcal{A} is called a k -lower triangular block tensor, or simply a lower triangular block tensor.

Definition 20 Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\alpha \subseteq [n]$. A principal subtensor $\mathcal{A}[\alpha]$ of the tensor \mathcal{A} is defined as an order m dimensional $|\alpha|$ tensor with entries

$$\mathcal{A}[\alpha] = (a_{i_1 \dots i_m}), \quad i_1, \dots, i_m \in \alpha.$$

Definition 21 (Shao et al. 2013) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$, and n_1, \dots, n_r be positive integers with $n_1 + \dots + n_r = n$ ($r \geq 1$). Let

$$I_1 = \{1, 2, \dots, n_1\},$$

$$I_i = \left\{ \left(\sum_{j \in [i-1]} n_j \right) + 1, \dots, \sum_{j \in [i]} n_j \right\} \subseteq [n] \quad (i \in [r] \setminus \{1\}),$$

and write $\mathcal{A}[I_i] = \mathcal{A}_i$. Suppose that for each $1 \leq i \leq r - 1$, the subtensor $\mathcal{A}[I_i \cup \dots \cup I_r]$ is a n_i -lower triangular block tensor, then \mathcal{A} is called a (n_1, \dots, n_r) -lower triangular block tensor with the diagonal blocks $\mathcal{A}_1, \dots, \mathcal{A}_r$.

Lemma 5 (Shao et al. 2013) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $m \geq 2$. Then there exists positive integers $r \geq 1$ and n_1, \dots, n_r with $n_1 + \dots + n_r = n$ ($r \geq 1$) such that \mathcal{A} is permutational similar to some $(n_1 + \dots + n_r)$ -lower triangular block tensor, where all the diagonal blocks $\mathcal{A}_1, \dots, \mathcal{A}_r$ are weakly irreducible.

Lemma 6 (Shao et al. 2013) Let $r \geq 2$ and n_1, \dots, n_r be positive integers with $n_1 + \dots + n_r = n$. Let \mathcal{A} be a $(n_1 + \dots + n_r)$ -lower triangular block tensor with the diagonal blocks $\mathcal{A}_1, \dots, \mathcal{A}_r$. Then we have:

$$\text{Det}(\mathcal{A}) = \prod_{i \in [r]} (\text{Det} \mathcal{A}_i)^{(m-1)^{n-n_i}},$$

and thus

$$\phi_{\mathcal{A}}(\lambda) = \prod_{i \in [r]} (\phi_{\mathcal{A}_i}(\lambda))^{(m-1)^{n-n_i}},$$

where $\phi_{\mathcal{A}}(\lambda)$ is the characteristic polynomial of the tensor \mathcal{A} .

From Theorem 1, Lemmas 5 and 6, we can easily obtain the following theorems.

Theorem 8 *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$, then there exists a permutation π on $[n]$ such that $P_\pi(\mathcal{A})$ is a $(n_1 + \dots + n_r)$ -lower triangular block tensor, where all the diagonal blocks $\mathcal{A}_1, \dots, \mathcal{A}_r$ are weakly irreducible.*

Remark 5 Theorem 8 could be called the “canonical form” theorem and the $(n_1 + \dots + n_r)$ -lower triangular block tensor is called the “canonical form” of \mathcal{A} .

Theorem 9 *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\bar{\mathcal{A}}$ be the “canonical form” of \mathcal{A} . Then*

$$\sigma(\mathcal{A}) = \sigma(\bar{\mathcal{A}}) = \bigcup_{i \in [r]} \sigma(\mathcal{A}_i).$$

Proof From Theorem 8 and (vi) of Theorem 3, we have

$$\sigma(\mathcal{A}) = \sigma(\bar{\mathcal{A}}).$$

From Lemma 6, we have

$$\sigma(\bar{\mathcal{A}}) = \bigcup_{i \in [r]} \sigma(\mathcal{A}_i).$$

Therefore, the conclusion holds. □

Canonical form of tensors plays an important role in decomposition of symmetric tensors as a sum of component rank-one tensors, which has numerous applications in electrical engineering and higher order statistics (Brachat et al. 2009; Kolda and Bader 2009; Robeva 2016), such Independent Component Analysis (Comon 1992). Given a symmetric tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, the aim is to decompose it as

$$\mathcal{A} = \sum_{i \in [k]} \lambda_i v_i^{\otimes m}, \tag{4}$$

where $v_1, \dots, v_k \in \mathbb{R}^n$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. Comon et al. (2008) showed that the decomposition of the form (4) exists. Furthermore, in Brachat et al. (2009) an algorithm for decomposition a symmetric tensor into a sum of component rank-one tensors was presented. However, for a tensor with large order or dimension the algorithm may result in large computing quantity. According to Theorem 8, and taking into account that \mathcal{A} is symmetric, there exists a permutation π on $[n]$ such that $P_\pi(\mathcal{A})$ is a diagonal block tensor, where all the diagonal blocks $\mathcal{A}_1 \in \mathbb{R}^{[m,n_1]}, \dots, \mathcal{A}_r \in \mathbb{R}^{[m,n_r]}$ ($\sum_{i \in [r]} n_i = n$) are weakly irreducible. Therefore, we can use algorithm in Brachat et al. (2009) decomposition of lower dimensional symmetric tensor $\mathcal{A}_j, j \in [r]$ as follows

$$\mathcal{A}_j = \sum_{i \in [k_j]} \lambda_{ji} v_{ji}^{\otimes m}, \quad j \in [r], v_{ji} \in \mathbb{R}^{n_j}.$$

Therefore,

$$P_\pi(\mathcal{A}) = \sum_{j \in [r]} \sum_{i \in [k_j]} \lambda_{ji} \omega_{ji}^{\otimes m},$$

where

$$\omega_{ji} = (\underbrace{0, \dots, 0}_{n_1 \text{ times}}, \dots, \underbrace{0, \dots, 0}_{n_{j-1} \text{ times}}, (v_{ji})^T, \underbrace{0, \dots, 0}_{n_{j+1} \text{ times}}, \dots, \underbrace{0, \dots, 0}_{n_r \text{ times}})^T \in \mathbb{R}^n, j \in [r], i \in [k_j],$$

thus, we obtain decomposition of the tensor \mathcal{A} as follows.

$$\mathcal{A} = \sum_{j \in [r]} \sum_{i \in [k_j]} \lambda_{ji} \left(P_\pi^{-1}(\omega_{ji}) \right)^{\otimes m}.$$

We shall employ an example to illustrate this interesting property.

Example 1 Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,4]}$ be a symmetric tensor defined by

$$\begin{aligned} a_{1111} &= 3, a_{2222} = 15, a_{3333} = 18, a_{1333} = a_{3133} = a_{3313} = a_{3331} = 6, \\ a_{1133} &= a_{1313} = a_{1331} = a_{3113} = a_{3131} = a_{3311} = 6, \\ a_{2224} &= a_{2242} = a_{2422} = a_{4222} = 9, a_{2444} = a_{4244} = a_{4424} = a_{4442} = 3, \\ a_{2244} &= a_{2424} = a_{2442} = a_{4224} = a_{4242} = a_{4422} = 3, \text{ other, } a_{i_1 i_2 i_3 i_4} = 0. \end{aligned}$$

By Theorem 8, we can obtain the canonical form of \mathcal{A} as follows. Let π be a permutation on $\{1, 2, 3, 4\}$, where $\pi(1) = 1, \pi(2) = 3, \pi(3) = 2, \pi(4) = 4$. Therefore, $P_\pi(\mathcal{A}) = (b_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,4]}$ reads as follows

$$\begin{aligned} b_{1111} &= 3, b_{2222} = 18, b_{3333} = 15, b_{1222} = b_{2122} = b_{2212} = b_{2221} = 6, \\ b_{1122} &= b_{1212} = b_{1221} = b_{2112} = b_{2121} = b_{2211} = 6, \\ b_{3334} &= b_{3343} = b_{3433} = b_{4333} = 9, b_{3444} = b_{4344} = b_{4434} = b_{4443} = 3, \\ b_{3344} &= b_{3434} = b_{3443} = b_{4334} = b_{4343} = b_{4433} = 3, \text{ other, } b_{i_1 i_2 i_3 i_4} = 0. \end{aligned}$$

It is easy to obtain that $P_\pi(\mathcal{A})$ is a diagonal block tensor with the diagonal blocks $P_\pi(\mathcal{A})[I_j], j \in \{1, 2\}$, where $I_1 = \{1, 2\}, I_2 = \{3, 4\}$. By algorithm 1 in Brachat et al. (2009), we obtain the rank-one decomposition of the diagonal block tensors $P_\pi(\mathcal{A})[I_j], j \in \{1, 2\}$ as follows

$$\begin{aligned} P_\pi(\mathcal{A})[I_1] &= v_{11}^{\otimes 4} + 2v_{12}^{\otimes 4}, \\ P_\pi(\mathcal{A})[I_2] &= v_{21}^{\otimes 4} - v_{22}^{\otimes 4}, \end{aligned}$$

where

$$v_{11} = (1, 2)^T, \quad v_{12} = (1, -1)^T, \quad v_{21} = (2, 1)^T, \quad v_{22} = (1, -1)^T.$$

Therefore

$$P_\pi(\mathcal{A}) = \omega_{11}^{\otimes 4} + 2\omega_{12}^{\otimes 4} + \omega_{21}^{\otimes 4} - \omega_{22}^{\otimes 4},$$

where

$$\begin{aligned}\omega_{11} &= (v_{11}^T, 0, 0)^T = (1, 2, 0, 0)^T, & \omega_{12} &= (v_{12}^T, 0, 0)^T = (1, -1, 0, 0)^T, \\ \omega_{21} &= (0, 0, v_{21}^T)^T = (0, 0, 2, 1)^T, & \omega_{22} &= (0, 0, v_{22}^T)^T = (0, 0, 1, -1)^T.\end{aligned}$$

which implies the rank-one decomposition of tensor \mathcal{A} given by

$$\mathcal{A} = (P_{\pi}^{-1}(\omega_{11}))^{\otimes 4} + 2(P_{\pi}^{-1}(\omega_{12}))^{\otimes 4} + (P_{\pi}^{-1}(\omega_{21}))^{\otimes 4} - (P_{\pi}^{-1}(\omega_{22}))^{\otimes 4},$$

where,

$$\begin{aligned}P_{\pi}^{-1}(\omega_{11}) &= (1, 0, 2, 0)^T, & P_{\pi}^{-1}(\omega_{12}) &= (1, 0, -1, 0)^T, \\ P_{\pi}^{-1}(\omega_{21}) &= (0, 2, 0, 1)^T, & P_{\pi}^{-1}(\omega_{22}) &= (0, 1, 0, -1)^T.\end{aligned}$$

Conclusions

In this paper, the definition of permutation transformation of tensors is introduced, and its properties are studied. As applications of permutation transformations, we give the canonical form theorem of tensors and a numerical example which show that some problems of higher dimension tensors can be translated into the corresponding problems of lower dimension weakly irreducible tensors by using the permutation transformation. There are many problems unsolved for permutation transformations of tensors and their applications. Hence, this paper is only a starting point for studying permutation transformations of tensors.

Authors' contributions

YTL, ZBL, QLL and QL are contributed equally to this work. All authors read and approved the final manuscript.

Acknowledgements

The authors are grateful to the anonymous referees for their valuable comments, which helped improve the quality of the paper. This work was supported by the National Nature Science Foundation of China (Grant No. 11361074).

Competing interests

The authors declare that they have no competing interests.

Received: 12 April 2016 Accepted: 21 November 2016

Published online: 28 November 2016

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