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The valuation of currency options by fractional Brownian motion

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Abstract

This research aims to investigate a model for pricing of currency options in which value governed by the fractional Brownian motion model (*FBM*). The fractional partial differential equation and some Greeks are also obtained. In addition, some properties of our pricing formula and simulation studies are presented, which demonstrate that the *FBM* model is easy to use.

Keywords: Black–Scholes model, Fractional Brownian motion, Currency option, Option pricing

Background

A currency options refers to an agreement that gives right to the holder in order to buy or sell a defined amount of foreign currency at a constant exercise price on option exercise. American options are traded at any time before they expire. European options can be exercised only during a specified period immediately before expiration.

Black and Scholes (1973) put forward option pricing in 1973, which leads to be studied by different scholars (Dravid et al. 1993; Toft and Reiner 1997; Kwok 2000; Duan and Wei 1999) claim that two issues in stock markets are not able to be presented clearly in this option pricing introduced by *BS* in accordance with Brownian motion (*BM*). These concepts refer to asymmetric leptokurtic features and the volatility smile. In view of this, the *BS* model was improved by Garman and Kohlhagen (1983) in order to assess European currency options by considering two prominent features;

1. The market volatility estimation of an underlying as obvious as price and time functioning void of referring to the characteristics of a particular investor directly. These characteristics could be functions of utility, measures of risk aversion, or yield expecting.
2. Strategy of self-replicating or hedging.

However, it is significant to note that the mispriced currency options by the *G–K* model were also substantiated in some studies (Cookson 1992). The most important reason of inappropriateness of this model for stock markets is the fact that the currencies are different from stocks so that the currency behavior is not captured by geometric Brownian motion (Ekvall et al. 1997). To tackle this problem regarding pricing currency options,

various models were recommended by modifying the G – K model (Rosenberg 1998; Sarwar and Krehbiel 2000; Bollen and Rasiel 2003; Shokrollahi and Kılıçman 2014a, b, 2015).

In view of this, the independency of logarithmic returns of the exchange rate was pointed out in all these studies along with the distribution of normal random variables. In addition, the empirical studies reveal that the logarithmic returns disseminations in the asset markets widely manifest excess kurtosis with high possibility of mass around the origin and in the tails, and indicate low possibility in the flanks in comparison with normal distribution of data. It means that financial return series include the properties, which are not normal, independent, linear and are self-similar, with heavy tails. Both autocorrelations and cross-correlations and also volatility clustering are considered to these properties.

In this regard, two fundamental features are considered in FBM namely self-similarity and long-range dependence. Then, employing this process is more feasible in terms of capturing the behavior from financial asset (Carbone et al. 2004; Wang et al. 2010). Although, FBM is neither a semi-martingale nor a Markov process then, we are not able to employ the conventional stochastic calculus for analyzing it. Fortunately, the research interest in this field was re-encouraged by new insights in stochastic analysis based on the Wick integration (see Hu and Øksendal 2003) called the fractional-Ito-integral. Using this type of stochastic integration (Hu and Øksendal 2003) proofed that the fractional Black–Scholes market presents no arbitrage opportunity and is complete. However, Björk and Hult (2005) argued that the use of FBM in this context does not make much economic sense because, while Wick integration leads to no arbitrage, the definition of the corresponding self-financing trading strategies is quite restrictive and, for example, in the setup of Elliott and Van der Hoek (2003), the simple buy-and-hold strategy is not self-financing. We noted that this arbitrage example in discrete-time does not, however, rule out the use of FBM in finance. For example, Bender et al. (2007) showed that the existence of arbitrage opportunities depends very much on the definition of the admissible trading strategies. Furthermore, Bender et al. (2008) stated that the financial market does not admit arbitrage opportunities in a class of trading strategies if a continuous price process has the conditional small ball property and pathwise quadratic variation. Hence it is not too hard to accept this idea: some restrictions are sufficient to exclude arbitrage in the fractional Brownian market. Indeed, some authors have used the geometric FBM to capture the behavior of underlying asset and to obtain fractional Black–Scholes formulas for pricing options, including Necula (2002) and Bayraktar et al. (2004).

In this paper, the pricing formula is investigated for pricing currency options by using the FBM model. Furthermore, we obtain risk neutral valuation model and fractional Black–Scholes equation. Some properties and numerical studies of our pricing formula are also analyzed. “Preparations” section deals with the definition and features of the FBM process, and some results regarding quasi-conditional expectation are also investigated. In “Pricing model” section, option pricing formula for the European currency options is derived by the FBM model. “Properties of pricing formula” section describe the fractional differential equation and also investigates some Greeks of our model. We show empirical studies and simulation in “Numerical studies” section in order to indicate the efficiency of the FBM model and final section of the paper is “Conclusion”.

Preparations

This section deals with some assumptions and definitions which is needed for this study. For get more information you can see Necula (2002), Cheridito (2003), Mishura (2008), and Hu and Øksendal (2003).

Definition 1 A FBM, $B_H(t)$ with Hurst exponential $H \in (0, 1)$ under the probability space (Ω, F, P) is a continuous Gaussian process with these features:

1. $B_H(0) = 0$.
2. $E[B_H(t)] = 0$ for all $t \geq 0$.
3. $\text{cov}[B_H(t)B_H(s)] = \frac{1}{2}[t^{2H} + s^{2H} - |t - s|^{2H}]$ for all $s, t \geq 0$.
4. If $H = \frac{1}{2}$ the $B_H(t)$ is equivalent to the Brownian motion.

Moreover, $E(B_H(t) - B_H(s))^2 = |t - s|^{2H}$ and $B_H(t)$ is stationary increments and is H -self-similar in the sense that $B_H(ct)$ and $c^H B_H(t)$ have the similar distribution for every $c > 0$. If $H > \frac{1}{2}$ the process $B_H(t)$ represents long-range correlation, by the following definition:

$$\sum_{m=1}^{\infty} E[B_H(1)(B_H(m+1) - B_H(m))] = \infty. \quad (1)$$

Now, suppose (Ω, F, P) be a probability field such that B_t^H is a FBM with respect to P , Some results represented that is required for the following (see Necula 2002).

Lemma 2 Consider the fractional differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H \quad S_0 = S, \quad (2)$$

then

$$S_t = S_0 \exp \left(\mu t + \sigma B_t^H - \frac{1}{2} \sigma^2 t^{2H} \right). \quad (3)$$

Lemma 3 Let $0 < t < T$ and $\sigma \in \mathbb{C}$ then

$$\tilde{E}_t \left[e^{\sigma B_T^H} \right] = e^{\sigma B_t^H + \frac{\sigma^2}{2} (T^{2H} - t^{2H})}, \quad (4)$$

where \tilde{E}_t shows the quasi-conditional expectation under risk-neutral measure.

Lemma 4 Suppose f be a function such that $\tilde{E}_t[f(B_T^H)] < \infty$. Thus for each $0 < t \leq T$ and $\sigma \in \mathbb{C}$, we have

$$\tilde{E}_t[f(\sigma B_T^H)] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2(T^{2H} - t^{2H})}} \times \exp \left[-\frac{(x - \sigma B_t^H)^2}{2\sigma^2(T^{2H} - t^{2H})} \right] f(x) dx. \quad (5)$$

Let $f(x) = \mathbf{1}_A$ thus, the following corollary is obtained.

Corollary 5 Assume $A \in B(R)$. Therefore

$$\tilde{E}_t[\mathbf{1}_A(\sigma B_T^H)] = \int_R \frac{1}{\sqrt{2\pi\sigma^2(T^{2H} - t^{2H})}} \times \exp\left[-\frac{(x - \sigma B_t^H)^2}{2\sigma^2(T^{2H} - t^{2H})}\right] \mathbf{1}_A(x) dx. \quad (6)$$

Assume $\theta, w \in R$. Then, this process considered

$$Z_t^* = \theta(B_t^H)^* = \theta B_t^H + \theta^{2H}, \quad 0 \leq t \leq T. \quad (7)$$

According to the Girsanov formula, there is a measure P^* such that Z_t^* is a new FBM. We will denote $E_t^*[\cdot]$ is a quasi-conditional expectation under P^* . Consider

$$X_t = \exp\left(-\theta B_t^H - \frac{\theta^2}{2} t^{2H}\right). \quad (8)$$

Lemma 6 Let f be a function such that $\tilde{E}_t[f(\theta B_t^H)] \leq \infty$. Thus for each $t \leq T$,

$$\tilde{E}_t^*[f(\theta B_T^H)] = \frac{1}{X_t} \tilde{E}_t[f(\theta B_T^H) X_T]. \quad (9)$$

Theorem 7 The price at every time $t \in [0, T]$ of a bounded F_T^H -measurable claim $F \in L^2$ as follows

$$F_t = e^{-r(T-t)} \tilde{E}_t[F], \quad (10)$$

where r shows the fixed rate of riskless interest.

Pricing model

Since, the system in finance is considered as an intricate system in investments in which investors avoid to make instant decisions after obtaining financial information in a fractional system. It means that achieving information to its threshold limit value is the major criteria for making decisions of investors rather than financial information with high flexibility. The asymmetric leptokurtic and long memory properties result from this behavior. In this regard, the beneficial model seems to be FBM model.

To derive the new currency option pricing formula in a fractional market. The following hypothesis will be provided:

1. there are no transaction costs or taxes;
2. security trading is continuous;
3. The rate of domestic interest r_d and the rate of foreign interest r_f are known and fixed throughout time;
4. There are no riskfree arbitrage opportunities.

Now, we consider a fractional Black–Scholes currency market that has two investments:

(a) a money market account

$$dM_t = r_d M_t dt, \quad (11)$$

where r_d show the rate of domestic interest.

(b) a stock whose price satisfies the following equation:

$$dS_t = \mu S_t + \sigma S_t d\hat{B}_t^H \quad 0 < t \leq T \quad S_0 = S > 0, \quad (12)$$

where $\frac{1}{2} < H < 1$ is Hurst parameter.

Let $B_t^H = \frac{\mu + r_f - r_d}{\sigma} t + \hat{B}_t^H$, hence respect to risk-neutral measure we have:

$$dS_t = (r_d - r_f) S_t + \sigma S_t dB_t^H \quad 0 < t \leq T \quad S_0 = S > 0. \quad (13)$$

Then, the solution for Eq. (13) is

$$S_t = S_0 \exp \left((r_d - r_f)t + \sigma B_t^H - \frac{1}{2} \sigma^2 t^{2H} \right). \quad (14)$$

Theorem 8 *The value at every $t \in [0, T]$ of a European call currency option with exercise price K and expiration T is given by*

$$C(t, S_t) = S_t e^{-r_f(T-t)} \Phi(d_1) - K e^{-r_d(T-t)} \Phi(d_2), \quad (15)$$

where

$$d_1 = \frac{\ln \left(\frac{S_t}{K} \right) + (r_d - r_f)(T - t) + \frac{\sigma^2}{2} (T^{2H} - t^{2H})}{\sigma \sqrt{T^{2H} - t^{2H}}}$$

$$d_2 = \sigma \sqrt{T^{2H} - t^{2H}}. \quad (16)$$

Corollary 9 *The value of European put currency option is given by*

$$P(t, S_t) = K e^{-r_d(T-t)} \Phi(-d_2) - S_t e^{-r_f(T-t)} \Phi(-d_1), \quad (17)$$

where

$$d_1 = \frac{\ln \left(\frac{S_t}{K} \right) + (r_d - r_f)(T - t) + \frac{\sigma^2}{2} (T^{2H} - t^{2H})}{\sigma \sqrt{T^{2H} - t^{2H}}}$$

$$d_2 = \sigma \sqrt{T^{2H} - t^{2H}}. \quad (18)$$

Properties of pricing formula

Assume that V is the value of currency options which depends just on t and S_t . Thus, the value of whole portfolio satisfies in the partial differential equation that present in this theorem.

Theorem 10 *The value of a currency options $V(t, S_t)$ satisfies in the following PDE*

$$\frac{\partial V}{\partial t} + H \sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r_d - r_f) S_t \frac{\partial V}{\partial S_t} - r_d V = 0. \quad (19)$$

Now, we discuss the properties of the FBM model such as Greeks, which summarize how option prices change with respect to underlying variables that are critically important in asset pricing and risk management. In addition, it can be used to rebalance the portfolio to achieve desired exposure to a certain risk. It is significant to note that, knowing the Greek, a particular exposure can be hedged from adverse changes in the market by employing the appropriate amount of other related financial instruments. Contrary to option prices, observed in the market, Greeks can not be found and have to be calculated by a model assumption. Typically, the Greeks are computed using a partial differentiation of the price formula Shokrollahi et al. (2015, 2016).

Theorem 11 *The Greeks can be written as*

$$\Delta = \frac{\partial C}{\partial S_t} = e^{-r_f(T-t)} \Phi(d_1), \quad (20)$$

$$\nabla = \frac{\partial C}{\partial K} = -e^{-r_d(T-t)} \Phi(d_2), \quad (21)$$

$$\rho_{r_d} = \frac{\partial C}{\partial r_d} = K(T-t)e^{-r_d(T-t)} \Phi(d_2), \quad (22)$$

$$\rho_{r_f} = \frac{\partial C}{\partial r_f} = S_t(T-t)e^{-r_f(T-t)} \Phi(d_1), \quad (23)$$

$$\begin{aligned} \Theta = \frac{\partial C}{\partial t} &= S_t r_f e^{-r_f(T-t)} \Phi(d_1) - K r_d e^{-r_d(T-t)} \Phi(d_2) \\ &\quad - S_t e^{-r_f(T-t)} \frac{\sigma H t^{2H-1}}{\sqrt{T^{2H} - t^{2H}}} \Phi'(d_1), \end{aligned} \quad (24)$$

$$\Gamma = \frac{\partial^2 C}{\partial S_t^2} = e^{-r_f(T-t)} \frac{\Phi'(d_1)}{S_t \sqrt{\sigma^2 (T^{2H} - t^{2H})}}, \quad (25)$$

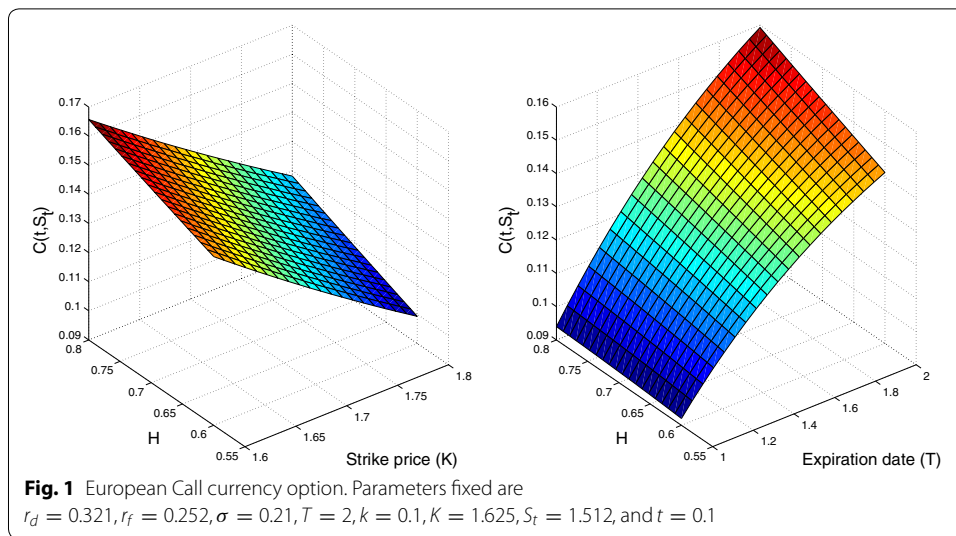
$$\vartheta_\sigma = \frac{\partial C}{\partial \sigma} = S_t e^{-r_f(T-t)} \sqrt{T^{2H} - t^{2H}} \Phi'(d_1). \quad (26)$$

The Hurst parameter H play a significant role in the FBM model. Then, we represents the influence of this parameter in the following theorem.

Theorem 12 *The impact of the Hurst parameter as follows*

$$\frac{\partial C}{\partial H} = S_t e^{-r_f(T-t)} \frac{\sigma (T^{2H} \ln T - t^{2H} \ln t)}{\sqrt{T^{2H} - t^{2H}}} \Phi'(d_1). \quad (27)$$

Fig. 1 shows the impact of parameters on our pricing formula.



The following theorem presents the estimation of volatility by R / S method.

Theorem 13 Assume $0 \leq T_1 < T_2$ be given, and let a partition of this interval is chosen, $T_1 = t_0 < t_1 < \dots < t_n = T_2$. Suppose S_{t_i} show the time series of observed price. Thus, the volatility of interval $[T_1, T_2]$ is

$$\sigma^2 = \frac{1}{T_2^H - T_1^H} \sum_{j=0}^{n-1} \left(\log \frac{S_{t_{j+1}}}{S_{t_j}} \right)^2. \quad (28)$$

Remark 14 The relationship of call-put parity is given by

$$C(t, S_t) - P(t, S_t) = S_t e^{-r_f(T-t)} - K e^{-r_d(T-t)}. \quad (29)$$

Remark 15 The relationship of put-call parity satisfies

$$\frac{\partial C(t, S_t)}{\partial S_t} - \frac{\partial P(t, S_t)}{\partial S_t} = e^{-r_f(T-t)}. \quad (30)$$

Remark 16 The delta of spot exercise price has a space-homogeneity feature, such that for every $b > 0$,

$$bC(t, S_t) = bS_t e^{-r_f(T-t)} \Phi(d_1) - bK e^{-r_d(T-t)} \Phi(d_2), \quad (31)$$

and

$$bP(t, S_t) = bK e^{-r_d(T-t)} \Phi(-d_2) - bS_t e^{-r_f(T-t)} \Phi(-d_1). \quad (32)$$

Furthermore, differentiating both sides with under b and thus by $b = 1$ we have

$$C(t, S_t) = S_t \frac{\partial C(t, S_t)}{\partial S_t} + K \frac{\partial C(t, S_t)}{\partial K}, \quad (33)$$

and

$$P(t, S_t) = S_t \frac{\partial P(t, S_t)}{\partial S_t} + K \frac{\partial P(t, S_t)}{\partial K}. \quad (34)$$

In fact, these equation is other model of the pricing currency option, when the value of stock is measured in a various unit. Moreover, $C'_{S_t}(t, S_t)$, $C'_K(t, S_t)$, $P'_{S_t}(t, S_t)$ and $P'_K(t, S_t)$ can be obtained by comparing this model with Eqs. (15), (17). These methods gives a new model for calculate delta.

Numerical studies

This section deals with how implement the *FBM* model and shows the impact of Hurst parameter H . In the present study, we consider the real call currency options values from Philadelphia Stock exchange (*PHLX*) in order to investigate some information concerning our pricing formula. By applying the R/S method, we estimate the exponent parameter for EUR/USD and then we obtain $H = 0.6102$. Furthermore, the volatility estimation is obtained by utilizing the historical volatility as follows;

$$L_i = \ln \left(\frac{q_{i+1}}{q_i} \right), \quad (35)$$

$$\sigma = \sqrt{\frac{\sum (L_i - \bar{L})^2}{N - 1}}, \quad \bar{L} = \frac{1}{N} \sum L_i, \quad (36)$$

where q_i show the daily value of exchange rate.

These data are extracted from 01/06/2010 to 01/12/2010 (six months) with the following parameters:

$K = 1.35$, $\sigma = 0.1201$, $r_d = 0.0231$, $r_f = 0.0352$, $T = 0.5$, and $t = 0.1$. We use the MATLAB software for obtaining results by different models such as *G-K*, *BS* and *FBM* models. The values calculated by these models are represented in Table 1, where P_{Actual} indicates the price of call currency options from *PHLX*, and the P_{BS} is the values computed by the *BS* model. In addition, the P_{FBM} points to the values calculated by *FBM* model. According to Table 1 our findings are more consistent with the actual price

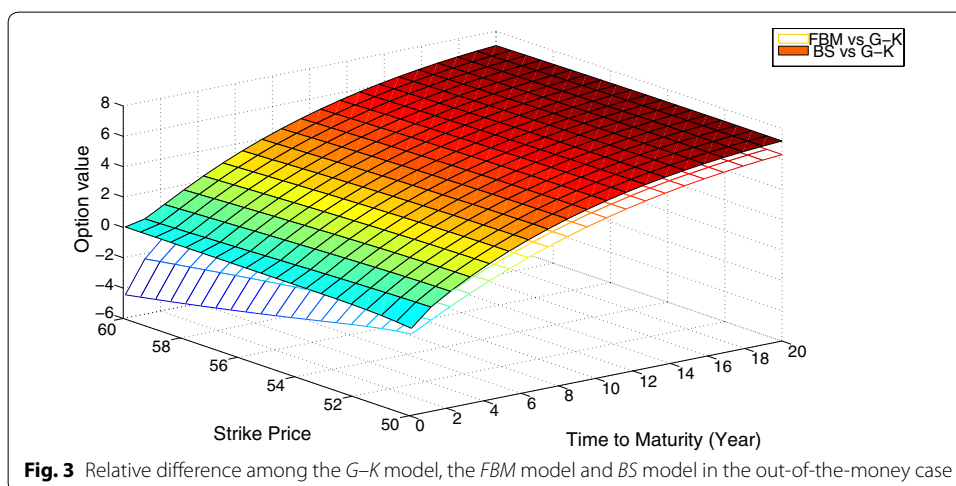
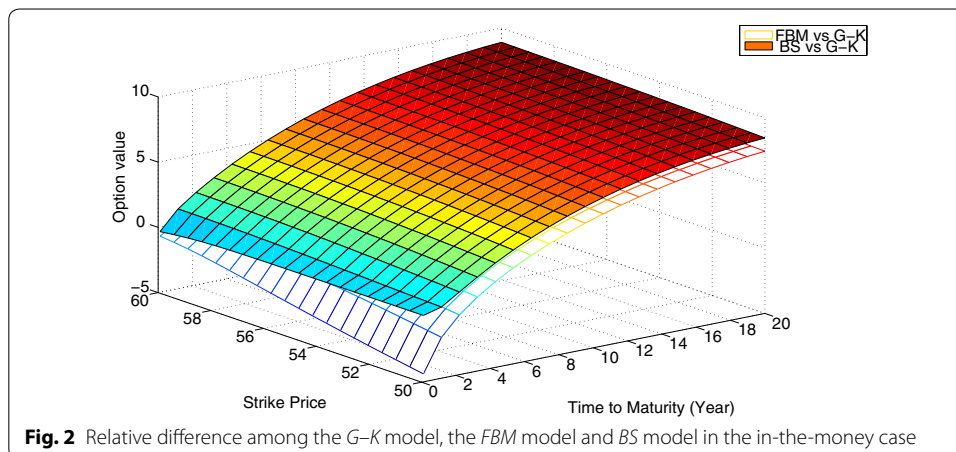
Table 1 Results by different pricing models

Exchange rate	P_{BS}	P_{FBM}	P_{Actual}
1.351	0.0377	0.0358	0.0338
1.357	0.0408	0.0388	0.0362
1.362	0.0433	0.0414	0.0391
1.368	0.0464	0.0444	0.0423
1.373	0.0490	0.0470	0.0456
1.379	0.0521	0.0501	0.0484
1.383	0.0542	0.0522	0.0503
1.389	0.0573	0.0553	0.0537
1.392	0.0589	0.0569	0.0548
1.398	0.0620	0.0601	0.0589

rather than the results of the other models. These properties reveal that our *FBM* model is able to get the behavior from financial market, which leads to creation of a satisfactory currency pricing model.

To further understand the preference of the *FBM* model, we calculated the theoretical prices of the our pricing formula and then we compare it with derived results from the *G-K* model and the *BS* model. For our propose, these parameter valuation are selected: $r_d = 0.0210$, $r_f = 0.0320$, $\sigma = 0.1050$, $t = 0.1$, $H = 0.78$, $S_t = 49$ for out-of-the-money case, $S_t = 61$ for in-the-money case with different exercise price $K \in [50, 60]$ and expiration date, $T \in [0.11, 20]$.

Figures 2 and 3 show the theoretical value discrepancy by the *G-K* model, *FBM* model and *BS* model, for in-the- money case and out-of-the-money case, respectively. These figures reveal that our pricing model are better matched with the *G-K* model. Then, from Table 1 and Figs. 2 and 3, we can conclude that our *FBM* model seems reasonable.



Conclusion

This study provided a new framework for pricing currency options in accordance with the *FBM* model to capture long-memory property of the spot exchange rate. In addition, a obtained a new formula for pricing European call currency options and the volatility estimation were presented. Some certain features and Greeks of currency options model are also obtained. Finally, we reported the empirical results for several models, which demonstrate that the *FBM* model would be reasonable.

Authors' contributions

All authors jointly worked on deriving the results. Both authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

Appendix

Proof of Theorem 8 In a risk neutral world, from Theorem 7 a European call currency option with maturity T and strike price K can be display as

$$\begin{aligned} C(t, S_t) &= \tilde{E}_t \left[e^{-r_d(T-t)} (S_T - K)^+ \right] \\ &= e^{-r_d(T-t)} \tilde{E}_t [S_T \mathbf{1}_{S_T > K}] - Ke^{-r_d(T-t)} \tilde{E}_t [\mathbf{1}_{S_T > K}]. \end{aligned} \quad (37)$$

We will first consider $\tilde{E}_t[\mathbf{1}_{S_T > K}]$. By setting

$$d_2^* = \ln \frac{K}{S} - (r_d - r_f)T + \frac{\sigma^2}{2} T^{2H}. \quad (38)$$

From Eq. (3), we have

$$S_t = S_0 \exp \left(\mu t + \sigma B_t^H - \frac{1}{2} \sigma^2 t^{2H} \right). \quad (39)$$

Then

$$\begin{aligned} \tilde{E}_t [\mathbf{1}_{S_T > K}] &= \tilde{E}_t [\mathbf{1}_{x > d_2^*} (\sigma B_t^H)] \\ &= \int_{d_2^*}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2(T^{2H} - t^{2H})}} \exp \left[-\frac{(x - \sigma B_t^H)^2}{2\sigma^2(T^{2H} - t^{2H})} \right] dx \\ &= \int_{\frac{d_2^* - \sigma B_t^H}{\sqrt{\sigma^2(T^{2H} - t^{2H})}}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= \int_{-\infty}^{\frac{\sigma B_t^H - d_2^*}{\sqrt{\sigma^2(T^{2H} - t^{2H})}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \Phi(d_2). \end{aligned} \quad (40)$$

where $z^2 = \frac{(x - \sigma B_t^H)^2}{\sigma^2(T^{2H} - t^{2H})}$, thus $x > d_2^*$ means that $z > \frac{d_2^* - \sigma B_t^H}{\sqrt{\sigma^2(T^{2H} - t^{2H})}}$ and the last equality follows since $\sigma B_t^H = \ln \frac{K}{S} - (r_d - r_f)t + \frac{\sigma^2}{2}t^{2H}$.

Now, we consider $\tilde{E}_t[S_T \mathbf{1}_{S_T > K}]$ setting

$$\sigma(B_t^H)^* = \sigma(B_t^H - \sigma t^{2H}). \quad (41)$$

Let

$$X_t = S \exp\left(\sigma B_t^H - \frac{1}{2}\sigma^2 t^{2H}\right). \quad (42)$$

Then we have $X_t = e^{-rt}S_t$. According to the Lemma 6, we obtain

$$\begin{aligned} \tilde{E}_t[S_T \mathbf{1}_{S_T > K}] &= e^{rt} \tilde{E}_t[X_T \mathbf{1}_{x > d_2^*}(\sigma B_T^H)] \\ &= e^{rt} X_t \tilde{E}_t^*[\mathbf{1}_{x > d_2^*}(\sigma B_T^H)] \\ &= e^{rt} X_t \tilde{E}_t^*[\mathbf{1}_{S_T > K}]. \end{aligned} \quad (43)$$

But

$$\begin{aligned} \ln S_T &= \ln S + (r_d - r_f)T + \sigma B_T^H - \frac{1}{2}\sigma^2 T^{2H} \\ &= \ln S + (r_d - r_f)T + \sigma(B_T^H)^* + \frac{1}{2}\sigma^2 T^{2H}. \end{aligned} \quad (44)$$

By setting $d_1^* = \ln \frac{K}{S} - (r_d - r_f)T - \frac{1}{2}\sigma^2 T^{2H}$, we obtain

$$\begin{aligned} \tilde{E}_t^*[\mathbf{1}_{S_T > K}] &= \tilde{E}_t[\mathbf{1}_{x > d_1^*}(\sigma(B_t^H))^*] \\ &= \int_{d_1^*}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2(T^{2H} - t^{2H})}} \exp\left[-\frac{(x - \sigma(B_t^H))^*^2}{2\sigma^2(T^{2H} - t^{2H})}\right] dx \\ &= \int_{\frac{d_1^* - \sigma(B_t^H)^*}{\sqrt{\sigma^2(T^{2H} - t^{2H})}}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= \int_{-\infty}^{\frac{\sigma B_t^H - d_2^*}{\sqrt{\sigma^2(T^{2H} - t^{2H})}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \Phi(d_1). \end{aligned} \quad (45)$$

The last equality follows since

$$\begin{aligned} \sigma B_t^H &= \ln \frac{S_t}{S} - (r_d - r_f)t + \frac{1}{2}\sigma^2 t^{2H} \\ \sigma(B_t^H)^* &= \sigma(B_t^H - \sigma t^{2H}). \end{aligned} \quad (46)$$

Then

$$\tilde{E}_t^*[\mathbf{1}_{S_T > K}] = e^{(r_d - r_f)T} X_t \Phi(d_1) = S_t e^{(r_d - r_f)(T-t)} \Phi(d_2). \quad (47)$$

□

Proof of Theorem 10 Let $V(t, S_t)$ be the price of the currency derivatives at time t and let Π be the portfolio value. Then we have

$$\Pi_t = V(S_t, t) - \Delta S_t. \quad (48)$$

Since

$$S_t = S_0 \exp \left[\mu T + \sigma B_T^H - \frac{1}{2} \sigma^2 T^{2H} \right]. \quad (49)$$

Then

$$\begin{aligned} D_u S_t &= S_t D_u \left(\mu \tau + \sigma B_\tau^H - \frac{1}{2} \sigma^2 \tau^{2H} \right) \\ &= S_t [D_u (\sigma B_\tau^H)], \\ D_u^\phi &= S_t H \sigma \tau^{2H-1}. \end{aligned} \quad (50)$$

Hence we have

$$\begin{aligned} d\Pi_t &= dV(t, S_t) - \Delta(dS_t + r_f S_t dt) \\ &= \left(\frac{\partial V}{\partial t} + H \sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \mu S_t \frac{\partial V}{\partial S_t} \right) dt \\ &\quad + \sigma S_t \frac{\partial V}{\partial S_t} dB_t^H - \Delta(\mu S_t dt + \sigma S_t dB_t^H + r_f S_t dt) \\ &= \left(\frac{\partial V}{\partial t} + H \sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \mu S_t \frac{\partial V}{\partial S_t} - \Delta \mu S_t - \Delta r_f S_t \right) dt \\ &\quad + \left(\sigma S_t \frac{\partial V}{\partial S_t} - \Delta \sigma S_t \right) dB_t^H. \end{aligned} \quad (51)$$

For eliminate the stochastic noise we choose $\Delta = \frac{\partial V}{\partial S_t}$, then

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + H \sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} - \Delta r_f S_t \right) dt. \quad (52)$$

The return of an amount Π_t invested in bank account equal to $r_d \Pi_t dt$ at time dt . For absence of arbitrage these values must be same, thus

$$\left(\frac{\partial V}{\partial t} + H \sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} - r_f S_t \frac{\partial V}{\partial S_t} \right) dt = r_d \Pi_t dt. \quad (53)$$

Since $\Pi_t = V(t, S_t) - \Delta S_t$, hence

$$\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} - r_f S_t \frac{\partial V}{\partial S_t} dt = r_d \left(V - S_t \frac{\partial V}{\partial S_t} \right), \quad (54)$$

so

$$\frac{\partial V}{\partial t} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r_d - r_f) S_t \frac{\partial V}{\partial S_t} - r_d V = 0. \quad (55)$$

□

Proof of Theorem 11 First, we derive a general formula. Let y be one of the influence factors. Thus we have

$$\begin{aligned} \frac{\partial C}{\partial y} &= \frac{\partial S_t e^{-(r_f)(T-t)}}{\partial y} \Phi(d_1) + S_t e^{-r_f(T-t)} \frac{\partial \Phi(d_1)}{\partial y} \\ &\quad - \frac{\partial K e^{-r_d(T-t)}}{\partial y} \Phi(d_2) - K e^{-r_d(T-t)} \frac{\partial \Phi(d_2)}{\partial y}. \end{aligned} \quad (56)$$

But

$$\begin{aligned} \frac{\partial \Phi(d_2)}{\partial y} &= \Phi'(d_2) \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{\left(d_1 - \sqrt{\sigma^2(T^{2H} - t^{2H})} \right)^2}{2} \right) \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \exp \left(d_1 \sqrt{\sigma^2(T^{2H} - t^{2H})} \right) \exp \left(-\frac{\sigma^2(T^{2H} - t^{2H})}{2} \right) \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \exp \left(\ln \frac{S_t}{K} + (r_d - r_f)(T - t) \right) \frac{\partial d_2}{\partial y} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{S}{K} \exp \left((r_d - r_f)(T - t) \right) \frac{\partial d_2}{\partial y}. \end{aligned} \quad (57)$$

Then we have that

$$\begin{aligned} \frac{\partial C}{\partial y} &= \frac{\partial S_t e^{-(r_f)(T-t)}}{\partial y} \Phi(d_1) - \frac{\partial K e^{-r_d(T-t)}}{\partial y} \Phi(d_2) \\ &\quad + S_t e^{-r_f(T-t)} \Phi'(d_1) \frac{\partial \sqrt{\sigma^2(T^{2H} - t^{2H})}}{\partial y}. \end{aligned} \quad (58)$$

Substituting in (58) we get the desired Greeks. □

Proof of Theorem 12

$$\begin{aligned}\eta &= \frac{\partial C}{\partial H} = S_t e^{-r_f(T-t)} \Phi'(d_1) \frac{\partial \sqrt{\sigma^2(T^{2H} - t^{2H})}}{\partial H} \\ &= S_t e^{-r_f(T-t)} \Phi'(d_1) \frac{\sigma(T^{2H} - t^{2H})}{\sqrt{T^{2H} - t^{2H}}}.\end{aligned}\quad (59)$$

□

Proof of Theorem 13 Since

$$S_t = S_0 \exp \left(\mu t + \sigma \widehat{B}_t^H - \frac{1}{2} \sigma^2 t^{2H} \right). \quad (60)$$

Then

$$\log \frac{S_{t_{j+1}}}{S_{t_j}} = \left(\mu(t_{j+1} - t_j) + \sigma(\widehat{B}_{t_{j+1}}^H - \widehat{B}_{t_j}^H) - \frac{1}{2} \sigma^2(t_{j+1}^2 - t_j^2) \right). \quad (61)$$

Hence the sum of the squares of the long return is

$$\sum_{j=0}^{n-1} \left(\log \frac{S_{t_{j+1}}}{S_{t_j}} \right)^2 = \left(\mu(t_{j+1} - t_j) + \sigma(\widehat{B}_{t_{j+1}}^H - \widehat{B}_{t_j}^H) - \frac{1}{2} \sigma^2(t_{j+1}^2 - t_j^2) \right)^2. \quad (62)$$

When the maximum step size $\|\Pi\| = \max_{j=0, \dots, n-1} (t_j - t_{j-1})$ is small. The right side of (27) is approximately equal to $\sigma^2(T_2^H - T_1^H)$ and then

$$\sigma^2 \approx \frac{1}{T_2^H - T_1^H} \sum_{j=0}^{n-1} \left(\log \frac{S_{t_{j+1}}}{S_{t_j}} \right)^2. \quad (63)$$

□

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