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Hybrid algorithm for common solution of monotone inclusion problem and fixed point problem and applications to variational inequalities

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Abstract
The aim of this paper is to investigate hybrid algorithm for a common zero point of the sum of two monotone operators which is also a fixed point of a family of countable quasi-nonexpansive mappings. We point out two incorrect proof in paper (Hecai in Fixed Point Theory Appl 2013:11, 2013). Further, we modify and generalize the results of Hecai's paper, in which only a quasi-nonexpansive mapping was considered. In addition, two family of countable quasi-nonexpansive mappings with uniform closeness examples are provided to demonstrate our results. Finally, the results are applied to variational inequalities.

Keywords: Quasi-nonexpansive mappings, Inverse-strongly monotone mapping, Maximal monotone operator, Fixed point

Mathematics Subject Classfication: 47H05, 47H09, 47H10

Introduction and preliminaries

The monotone inclusion problem is to

$$\text{find an} \quad x \in H \quad \text{such that} \quad 0 \in \sum_{i=1}^{m} A_i x,$$

where H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and A_i are set-valued maximal monotone operators (Hui and Lizhi 2013). Such problem is very important in many areas, such as convex optimization and monotone variational inequalities, for instance. There is an extensive literature to approach the inclusion problem, all of which can essentially be divided into two classes according to the number of operators involved: single operator class ($m = 1$) and multiple operator class ($m \geq 2$). The latter class can always be reduced to the case of $m = 2$ via Spingarn's method (Spingarn 1983). Based on a series of studies in the next decades, splitting methods for monotone operators were inspired and studied extensively. Splitting methods for linear equations were introduced by Peaceman and Rachford (1995) and Douglas and Rachford (1956). Extensions to nonlinear equations in Hilbert spaces were carried out by Kellogg (1969) and Lions and Mercier (1979).

The central problem is to iteratively find a zero of the sum of two monotone operators A and B in a Hilbert space H. Splitting methods have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as signal processing, image recovery and machine learning are mathematically modeled as a nonlinear operator equation (Shehu et al. 2016a, b; Shehu 2015). And the operator is decomposed into the sum of two nonlinear operators.

In this paper, we consider the problem of finding a solution for the following problem: find an x in the fixed point set of a family of countable quasi-nonexpansive mappings S_n such that

$$x \in (A+B)^{-1}(0),$$

where A and B are two monotone operators. The similar problem has been addressed by many authors in view of the applications in signal processing and image recovery; see, for example, Qin et al. (2010), Zhang (2012), Takahashi et al. (2010), Kamimura and Takahashi (2010) and the references therein.

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H, P_C be the metric projection from H onto C, and $S: C \to C$ be a mapping. We use $F(S)$ to denote the fixed point set of S_n below, i.e., $F(S) := \{x \in C : x = Sx\}$. Recall that S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall \quad x, y \in C.$$

If C is a bounded closed and convex subset of H, then $F(S)$ is nonempty closed and convex; see Browder (1976). S is said to be quasi-nonexpansive if $F(S) \neq \emptyset$ and

$$\|Sx - p\| \leq \|x - p\|, \quad \forall\, x \in C, \quad p \in F(S).$$

It is easy to see that nonexpansive mappings are Lipschitz continuous, however, the quasi-nonexpansive mapping is discontinuous on its domain generally. Indeed, the quasi-nonexpansive mapping is only continuous in its fixed point set.

Let $A: C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall\, x, y \in C.$$

A is said to be α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall\, x, y \in C.$$

A is said to be α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall\, x, y \in C.$$

Notice that, a α-inverse strongly monotone operator must be $\frac{1}{\alpha}$-Lipschitz continuous.

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall\, y \in C. \tag{1}$$

In this paper, we use $VI(C, A)$ to denote the solution set of (1). It is known that $x^* \in C$ is a solution to (1) if x^* is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant, I is the identity mapping, and P_C is the metric projection from H onto C. Next we recall some well-known definitions.

Definition 1 (*Takahashi et al.* 2010) A multi-valued operator $T : H \to H$ with the domain $D(T) = \{x \in H : Tx \neq 0\}$ and the range $R(T) = \{Tx : x \in D(T)\}$ is said to be monotone if for $x_1, x_2 \in D(T), y_1, y_2 \in R(T)$, the following inequality holds $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$.

Definition 2 (*Takahashi et al.* 2010) A monotone operator T is said to be maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator.

Definition 3 (*Takahashi et al.* 2010) Let I denote the identity operator on H and $T : H \to H$ be a maximal monotone operator. For each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda = (I - \lambda A)^{-1}$ is called the resolvent of T.

And it is known that $T^{-1}(0) = F(J_\lambda)$ for all $\lambda > 0$ and J_λ is firmly nonexpansive.

Three classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one was introduced in 1953 by Mann (1953) and is well known as Manns iteration process defined as follows:

$$\begin{cases} x_0 \quad \textit{chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \end{cases} \tag{2}$$

where the sequence $\{\alpha_n\}$ is chosen in [0,1]. Fourteen years later, Halpern (1967) proposed the new innovation iteration process which resembled Manns iteration (2). It is defined by

$$\begin{cases} x_0 \quad \textit{chosen arbitrarily}, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0, \end{cases} \tag{3}$$

where the element $u \in C$ is fixed. Seven years later, Ishikawa (1974) enlarged and improved Mann's iteration (2) to the new iteration method, which is often cited as Ishikawa's iteration process and defined recursively by

$$\begin{cases} x_0 \quad \textit{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \quad n \geq 0, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0,1].

Moreover, many authors have studied the common solution problem, that is, find a point in a solution set and a fixed (zero) point set of some nonlinear problems; see, for example, Kamimura and Takahashi (2000), Takahashi and Toyoda (2003), Ye and Huang (2011), Cho and Kang (2011), Zegeye and Shahzad (2012), Qin et al. (2010), Lu and Wang (2012), Husain and Gupta (2012), Noor and Huang (2007), Qin et al. (2009), Kim and Tuyen (2011), Wei and Shi (2012), Qin et al. (2010), Qin et al. (2008), He et al. (2011), Wu and Liu (2012), Qin and Su (2007), Abdel-Salam and Al-Khaled (2012), Qin et al. (2010), Zegeye et al. (2012) and the references therein. In Kamimura and Takahashi

(2000), in the framework of real Hilbert spaces, Kamimura and Takahashi investigated the problem of finding zero points of a maximal monotone operator by considering the following iterative algorithm:

$$x_0 \in H, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{\lambda_n} x_n, \quad n = 0, 1, 2, \cdots \tag{5}$$

where $\{\alpha_n\}$ is a sequence in (0,1), $\{\lambda_n\}$ is a positive sequence, $T : H \to H$ is a maximal monotone, and $J_{\lambda_n} = (I + \lambda_n T)^{-1}$. They showed that the sequence $\{x_n\}$ generated in (5) converges weakly to some $z \in T^{-1}(0)$ provided that the control sequence satisfies some restrictions. Further, using this result, they also investigated the case that $T = \partial f$, where $f : H \to H$ is a proper lower semicontinuous convex function.

Takahashi and Toyoda (2003) investigated the problem of finding a common solution of the variational inequality problem (1) and a fixed point problem involving nonexpansive mappings by considering the following iterative algorithm:

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) SP_C(x_n - \lambda_n A x_n), \quad \forall n \geq 0, \tag{6}$$

where $\{\alpha_n\}$ is a sequence in (0,1), $\{\lambda_n\}$ is a positive sequence, $S : C \to C$ is a nonexpansive mapping, and $A : C \to H$ is an inverse-strongly monotone mapping. They showed that the sequence $\{x_n\}$ generated in (6) converges weakly to some $z \in VI(C, A) \cap F(S)$ provided that the control sequence satisfies some restrictions.

Hecai (2013) studied the common solution for two monotone operators and a quasi-nonexpansive mapping in the framework of Hilbert spaces. The aim of this paper is to investigate hybrid algorithm for a common zero point of the sum of two monotone operators which is also a fixed point of a family of countable quasi-nonexpansive mappings. We point out two incorrect justifications in the proof of Theorem 2.1 in paper Hecai (2013). Further, we modify and generalize the results of Hecai's paper, in which only a quasi-nonexpansive mapping was considered. In addition, two family of countable quasi-nonexpansive mappings with uniform closeness examples are provided to demonstrate our results. Finally, we apply the results to variational inequalities.

To obtain our main results in this paper, we need the following lemmas and definitions.

Let C be a nonempty, closed, and convex subset of H. Let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a sequence of mappings of C into C such that $\cap_{n=1}^{\infty} F(S_n)$ is nonempty. Then $\{S_n\}_{n=1}^{\infty}$ is said to be *uniformly closed*, if $p \in \cap_{n=1}^{\infty} F(S_n)$, whenever $\{x_n\} \subset C$ converges strongly to p and $\|x_n - S_n x_n\| \to 0$ as $n \to \infty$.

Lemma 4 (Aoyama et al. 2007) *Let C be a nonempty, closed, and convex subset of H, $A : C \to H$ be a mapping, and $B : H \to 2^H$ be a maximal monotone operator. Then $F(J_r(I - \lambda A)) = (A + B)^{-1}(0)$.*

Let C be a nonempty, closed, and convex subset of H, the projection operator $P_C : E \to C$ is a map that assigns to an arbitrary point $x \in H$ the minimum point of the norm $\|x - y\|$, that is, $P_C x = \bar{x}$, where $\bar{x}$ is a unique solution to the minimization problem

$$\|\bar{x} - x\| = \min_{y \in C} \|y - x\|.$$

It is well-known that

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \quad \forall\, y \in C.$$

Abdel-Salam and Al-Khaled (2012) proved the following result.

Theorem 5 *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be an α -inverse-strongly monotone mapping, $S : C \to C$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and B be a maximal monotone operator on H such that the domain of B is included in C. Assume that $F = F(S) \cap (A + B)^{-1}(0) \neq \emptyset$. Let $\{\lambda_n\}$ be a positive real number sequence and $\{\alpha_n\}$ be a real number sequence in [0,1]. Let $\{x_n\}$ be a sequence of C generated by*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S J_{r_n}(x_n - \lambda_n A x_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$. Suppose that the sequences λ_n and α_n satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$;

(b) $0 < b \leq \lambda_n \leq c < 2\alpha$

Then the sequence $\{x_n\}$ converges strongly to $q = P_F x_0$.

However, the proof of above Theorem 5 is not correct. First mistake: in page 6, line 16–17, there is a mistake inequality:

$$\begin{aligned} \|z_n - p\|^2 &= \|J_{\lambda_n}(x_n - \lambda_n A x_n) - J_{\lambda_n}(p - \lambda_n A p)\|^2 \\ &\leq \langle (x_n - \lambda_n A x_n) - (p - \lambda_n A p), z_n - p \rangle. \end{aligned}$$

Second mistake: in page 7, -line 5–7, there is a mistake ratiocination:

Since B is monotone, we get for any $(u, v) \in B$ that

$$\left\langle z_n - u, \frac{x_n - z_n}{\lambda_n} - A x_n - v \right\rangle \geq 0. \tag{7}$$

Replacing n by n_i and letting $i \to \infty$, we obtain from (7) that

$$\langle \omega - u, -A\omega - v \rangle \geq 0.$$

Our comments: Notice that, the inner product $\langle \cdot, \cdot \rangle$ is not weakly continuous. For example: in Hilbert space l^2, let

$$\begin{aligned} x_0 &= (1, 0, 0, 0, 0, \ldots), \\ x_1 &= (1, 1, 0, 0, 0, \ldots), \\ x_2 &= (1, 0, 1, 0, 0, \ldots), \\ x_3 &= (1, 0, 0, 1, 0, \ldots), \\ &\cdots\cdots. \end{aligned}$$

It is well-known that $\{x_n\}$ converges weakly to x_0, but

$$\langle x_n, x_n \rangle = 2, \quad \langle x_0, x_0 \rangle = 1,$$

so the inner product $\langle x_n, x_n \rangle$ does not converges to $\langle x_0, x_0 \rangle$. Therefore,

$$\left\langle z_n - u, \frac{x_n - z_n}{\lambda_n} - Ax_n - v \right\rangle$$

does not converges to

$$\langle \omega - u, -A\omega - v \rangle.$$

In order to modify the iterative algorithm of Theorem 5 and to get more generalized results, we present a new iterative algorithm in this paper. Moreover, the results are applied to variational inequalities.

Main results

Now we are in the position to give our main results.

Theorem 6 *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be an α -inverse-strongly monotone mapping, and B be a maximal monotone operator on H such that the domain of B is included in C. Let $\{S_n\} : C \to C$ be a family of countable quasi-nonexpansive mappings which are uniformly closed. Assume that $F = F(S) \cap (A + B)^{-1}(0) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence and $\{\alpha_n\}$ be a real number sequence in [0,1). Let $\{x_n\}$ be a sequence of C generated by*

$$\begin{cases} x_1 \in C_1 = C, \quad \text{chosen} \quad \text{arbitrarily}, \\ z_n = J_{r_n}(x_n - r_n A x_n), \\ y_n = \alpha_n z_n + (1 - \alpha_n) S_n z_n, \\ C_{n+1} = \left\{ z \in C_n : \|z_n - z\| \leq \|y_n - z\| \leq \|x_n - z\| \right\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\liminf_{n\to\infty} r_n > 0$, $r_n \leq 2\alpha$ and $\limsup_{n\to\infty} \alpha_n < 1$. Then the sequence $\{x_n\}$ converges strongly to $q = P_F x_0$.

Proof We divide the proof into six steps.

Step 1. We show that C_n is closed and convex. Notice that $C_1 = C$ is closed and convex. Suppose that C_i is closed and convex for some $i \geq 1$. Next we show that C_{i+1} is closed and convex for the same i. Since

$$\begin{aligned} C_{i+1} &= C_i \cap \left\{ z \in E : \|y_i - z\| \leq \|z_i - z\| \right\} \cap \left\{ z \in E : \|z_i - z\| \leq \|x_i - z\| \right\} \\ &= C_i \cap \left\{ z \in E : \langle z, y_i - z_i \rangle \leq \frac{1}{2}\left(\|y_i\|^2 - \|z_i\|^2 \right) \right\} \\ &\quad \cap \left\{ z \in E : \langle z, z_i - x_i \rangle \leq \frac{1}{2}\left(\|z_i\|^2 - \|x_i\|^2 \right) \right\}. \end{aligned}$$

It is obvious that

$$\left\{z \in E : \langle z, y_i - z_i \rangle \leq \frac{1}{2}\left(\|y_i\|^2 - \|z_i\|^2\right)\right\},$$
$$\left\{z \in E : \langle z, z_i - x_i \rangle \leq \frac{1}{2}\left(\|z_i\|^2 - \|x_i\|^2\right)\right\}$$

are all closed and convex, so C_{i+1} is closed and convex. This shows that C_n is closed and convex for all $n \geq 1$.

Step 2. We show that $F \subset C_n$ for all $n \geq 1$. By the assumption, we see that $F \subset C_1$. Assume that $F \subset C_i$ for some $i \geq 1$. For any $p \in F \subset C_i$, we find from the Lemma that

$$p = S_i p = J_{r_i}(p - r_i Ap).$$

Since J_{r_i} is nonexpansive, we have

$$\begin{aligned}\|z_i - p\|^2 &= \|J_{r_i}(x_i - r_i Ax_i) - J_{r_i}(p - r_i Ap)\|^2 \\ &\leq \|(x_i - r_i Ax_i) - (p - r_i Ap)\|^2 \\ &= \|(x_i - p) - r_i(Ax_i - Ap)\|^2 \\ &= \|x_i - p\|^2 - 2r_i\langle x_i - p, Ax_i - Ap\rangle + r_i^2\|Ax_i - Ap\|^2 \\ &\leq \|x_i - p\|^2 - r_i(2\alpha - r_i)\|Ax_i - Ap\|^2,\end{aligned}$$

which implies that

$$\|z_i - p\| \leq \|x_i - p\|. \tag{8}$$

On the other hand, we have

$$\begin{aligned}\|y_i - p\| &= \|\alpha_i z_i + (1 - \alpha_i)S_i z_i - p\| \\ &= \|\alpha_i(z_i - p) + (1 - \alpha_i)(S_i z_i - p)\| \\ &\leq \alpha_i\|z_i - p\| + (1 - \alpha_i)\|S_i z_i - p\| \\ &\leq \alpha_i\|z_i - p\| + (1 - \alpha_i)\|z_i - p\| \\ &= \|z_i - p\|.\end{aligned} \tag{9}$$

From (8) and (9), we know that $p \in C_{i+1}$. This show $F \subset C_n$ for all $n \geq 1$.

Step 3. We show that $\{x_n\}$ is a Cauchy sequence, so it is convergent in C.

Since $x_n = P_{C_n}x_0$ and $C_{n+1} \subset C_n$, then we obtain

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \text{for all } n \geq 1. \tag{10}$$

Therefore $\|x_n - x_0\|$ is nondecreasing. On the other hand, we have

$$\|x_n - x_0\| = \|P_{C_n}x_0 - x_0\| \leq \|p - x_0\|,$$

for all $p \in F \subset C_n$ and for all $n \geq 1$. Therefore, $\|x_n - x_0\|$ is also bounded. This together with (10) implies that the limit of $\|x_n - x_0\|$ exists. Put

$$\lim_{n\to\infty} \|x_n - x_0\| = d. \tag{11}$$

It is known that for any positive integer m,

$$\begin{aligned}\|x_{n+m} - x_n\|^2 &= \|x_{n+m} - P_{C_n}x_0\|^2 \\ &\leq \|x_{n+m} - x_0\|^2 - \|P_{C_n}x_0 - x_0\|^2 \\ &= D_f(x_{n+m}, x_0) - D_f(x_n, x_0),\end{aligned}$$

for all $n \geq 1$. This together with (11) implies that

$$\lim_{n\to\infty} D_f(x_{n+m}, x_n) = 0,$$

uniformly for all m, holds. Therefore, we get that

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0,$$

uniformly for all m, holds. Then $\{x_n\}$ is a Cauchy sequence, hence there exists a point $p \in C$ such that $x_n \to p$.

Step 4. We prove that the limit of $\{x_n\}$ belongs to F.

Let $\lim_{n\to\infty} x_n = q$. Sine $x_{n+1} \in C_{n+1}$, so we have

$$\|y_n - x_{n+1}\| \leq \|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0, \tag{12}$$

as $n \to \infty$. Hence

$$\lim_{n\to\infty} y_n = q, \quad \lim_{n\to\infty} z_n = q. \tag{13}$$

From

$$y_n = \alpha_n z_n + (1 - \alpha_n) S_n z_n,$$

we have that

$$\|y_n - z_n\| = (1 - \alpha_n)\|S_n z_n - z_n\|.$$

The condition $\limsup_{n\to\infty} \alpha_n < 1$ and (13) imply that

$$\lim_{n\to\infty} \|S_n z_n - z_n\| = 0. \tag{14}$$

Because $\{S_n\}$ is an uniformly closed family of countable quasi-nonexpansive mappings, therefore this together with the (14) implies that $q \in \cap_{\infty}^{n=1} F(S_n)$.

Step 5. We show that $q \in (A + B)^{-1}(0)$.

Notice that $z_n = J_{r_n}(x_n - r_n A x_n)$. This means that

$$x_n - r_n A x_n \in z_n + r_n B z_n,$$

Actually, that is,

$$\frac{x_n - z_n}{r_n} - A x_n \in B z_n,$$

For B is monotone, so we get for any $(u, v) \in B$ that

$$\left\langle z_n - u, \frac{x_n - z_n}{r_n} - A x_n - v \right\rangle \geq 0. \tag{15}$$

Letting $n \to \infty$, we obtain from (15) that

$$\langle q - u, -Aq - v\rangle \geq 0.$$

Since B is a maximal monotone operator, so we have $-Aq \in Bq$, that is, $0 \in (A+B)(q)$. Hence, $q \in (A+B)^{-1}(0)$. This completes the proof that $q \in F$.

Step 6. We show that $q = P_F x_0$.

Observe that $P_F x_0 \in C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} x_0$, thus we have

$$\|x_{n+1} - x_0\| \leq \|P_F x_0 - x_0\|.$$

On the other hand, we have

$$\|x_0 - P_F x_0\| \leq \|x_0 - q\| = \lim_{n\to\infty} \|x_0 - x_{n+1}\| \leq \|x_0 - P_F x_0\|.$$

Since F is closed and convex, so the projection $P_F x_0$ is unique. Therefore we get that $q = P_F x_0$. This completes the proof. □

Application

In this section, we apply our results to variational inequalities.

Let $f : H \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. For all $x \in H$, define the subdifferential

$$\partial f(x) = \{z \in H : f(x) + \langle y - x, z\rangle \leq f(y), \quad \forall y \in H\}.$$

Then ∂f is a maximal monotone operator of H into itself (Noor and Huang 2007). Let C be a nonempty closed convex subset of H and i_C be the indicator function of C, that is,

$$i_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Furthermore, for any $v \in C$, we define the normal cone $N_C(v)$ of C at v as follows:

$$N_C v = \{z \in H : \langle z, y - v\rangle \leq 0, \quad \forall y \in H\}.$$

Then $i_C : H \to (-\infty,+\infty]$ is a proper lower semicontinuous convex function on H and ∂i_C is a maximal monotone operator. Let $Jx = (I + \lambda \partial i_C)^{-1} x$ for any $\lambda > 0$ and $x \in H$. From $\partial i_C x = N_C x$ and $x \in C$, we get

$$\begin{aligned} v = J_\lambda x &\Leftrightarrow x \in v + \lambda N_C v, \\ &\Leftrightarrow \langle x - v, y - v\rangle, \quad \forall y \in C, \\ &\Leftrightarrow v = P_C x, \end{aligned}$$

where P_C is the projection operator from H into C. In the same way, we can get that $x \in (A + \partial i_C)^{-1}(0) \Leftrightarrow x \in VI(A, C)$. Putting $B = \partial i_C$ in Theorem 6, we can see that $J_{\lambda_n} = P_C$. Naturally, we can obtain the following consequence.

Theorem 7 *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be an α -inverse-strongly monotone mapping, and $S_n : C \to C$ be a family of countable quasi-nonexpansive mappings which are uniformly closed. Assume that $F = F(S) \cap VI(C, A) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence and $\{\alpha_n\}$ be a real number sequence in [0,1). Let $\{x_n\}$ be a sequence of C generated by*

$$\begin{cases} x_1 \in C_1 = C, \quad chosen \quad arbitrarily, \\ z_n = P_C(x_n - r_n A x_n), \\ y_n = \alpha_n z_n + (1-\alpha_n) S_n z_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \le \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1, \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\liminf_{n\to\infty} r_n > 0$, $r_n \le 2\alpha$ *and* $\limsup_{n\to\infty} \alpha_n < 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $q = P_F x_0$.

Based on Theorem 7, we have the following corollary on variational inequalities.

Corollary 8 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, $A : C \to H$ *be an* α*-inverse-strongly monotone mapping. Assume that* $F = VI(C, A) \ne \emptyset$. Let $\{r_n\}$ be a positive real number *sequence. Let* $\{x_n\}$ *be a sequence of* C *generated by*

$$\begin{cases} x_1 \in C_1 = C, \quad chosen \quad arbitrarily, \\ z_n = P_C(x_n - r_n A x_n), \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1, \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$, *and* $\liminf_{n\to\infty} r_n > 0$, $r_n \le 2\alpha$. *Then the sequence* $\{x_n\}$ *converges strongly to* $q = P_{VI(C,A)} x_0$.

Examples

Let H be a Hilbert space and C be a nonempty closed convex and balanced subset of H. Let $\{x_n\}$ be a sequence in C such that $\|x_n\| = r > 0$, $\{x_n\}$ converges weakly to $x_0 \ne 0$ and $\|x_n - x_m\| \ge r > 0$ for all $n \ne m$. Define a family of countable mappings $\{T_n\} : C \to C$ as follows

$$T_n(x) = \begin{cases} \frac{n}{n+1} x_n & if\ x = x_n (\exists\, n \ge 1), \\ -x & if\ x \ne x_n (\forall\, n \ge 1). \end{cases}$$

Conclusion 9 $\{T_n\}$ *has a unique common fixed point* 0, *i.e.,* $F = \cap_{n=1}^{\infty} F(T_n) = \{0\}$, *for all* $n \ge 0$.

Proof The conclusion is obvious. □

Conclusion 10 $\{T_n\}$ *is a uniformly closed family of countable quasi-nonexpansive mappings.*

Proof First, we have

$$\|T_n x - 0\| = \begin{cases} \frac{n}{n+1} \|x_n - 0\|, & if\ x = x_n, \\ \|x - 0\| & if\ x \ne x_n. \end{cases}$$

Therefore

$$\|T_n x - 0\| \le \|x - 0\|^2,$$

for all $x \in C$. On the other hand, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - T_n z_n\| \to 0$ as $n \to \infty$, it is easy to see that there exists sufficiently large nature number N such that $z_n \neq x_m$, for any $n, m > N$. Then $Tz_n = -z_n$ for $n > N$. It follows from $\|z_n - T_n z_n\| \to 0$ that $2z_n \to 0$. Hence $z_n \to z_0 = 0$, that is $z_0 \in F$. □

Example 11 Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{\frac{1}{2}}, \quad \forall\, \xi \in l^2,$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n,\ \forall\, \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots),\ \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n \ldots) \in l^2.$$

Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \ldots), \\ x_1 &= (1, 1, 0, 0, \ldots), \\ x_2 &= (1, 0, 1, 0, 0, \ldots), \\ x_3 &= (1, 0, 0, 1, 0, 0, \ldots), \\ &\ldots\ldots\ldots\ldots \\ x_n &= (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \ldots, \xi_{n,k}, \ldots) \\ &\ldots\ldots\ldots\ldots, \end{aligned}$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1,\ n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1, \end{cases}$$

for all $n \geq 1$. It is well-known that $\|x_n\| = \sqrt{2}$, $\forall n \geq 1$ and $\{x_n\}$ converges weakly to x_0. Define a countable family of mappings $T_n : E \to E$ as follows

$$T_n(x) = \begin{cases} \frac{n}{n+1} x_n & \text{if } x = x_n, \\ -x & \text{if } x \neq x_n, \end{cases}$$

for all $n \geq 0$. By using Conclusion 9 and 10, $\{T_n\}$ is a uniformly closed family of countable quasi-nonexpansive mappings.

Example 12 Let $E = L^p[0,1]$ $(1 < p < +\infty)$ and

$$x_n = 1 - \frac{1}{2^n},\ n = 1, 2, 3, \cdots.$$

Define a sequence of functions in $L^p[0,1]$ as the following expression

$$f_n(x) = \begin{cases} \frac{2}{x_{n+1} - x_n} & \text{if} \quad x_n \leq x < \frac{x_{n+1} + x_n}{2}, \\ \frac{-2}{x_{n+1} - x_n} & \text{if} \quad \frac{x_{n+1} + x_n}{2} \leq x < x_{n+1} \\ 0 & \quad \text{otherwise} \end{cases}$$

for all $n \geq 1$. Firstly, we can see for any $x \in [0,1]$ that

$$\int_0^x f_n(t)dt \to 0 = \int_0^x f_0(t)dt, \tag{16}$$

where $f_0(x) \equiv 0$. It is well-known that the above relation (16) is equivalent to $\{f_n(x)\}$ converges weakly to $f_0(x)$ in uniformly smooth Banach space $L^p[0,1](1 < p < +\infty)$. On the other hand, for any $n \neq m$, we have

$$\begin{aligned}
\|f_n - f_m\| &= \left(\int_0^1 |f_n(x) - f_m(x)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_{x_n}^{x_{n+1}} |f_n(x) - f_m(x)|^p dx + \int_{x_m}^{x_{m+1}} |f_n(x) - f_m(x)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_{x_n}^{x_{n+1}} |f_n(x)|^p dx + \int_{x_m}^{x_{m+1}} |f_m(x)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\left(\frac{2}{x_{n+1} - x_n}\right)^p (x_{n+1} - x_n) + \left(\frac{2}{x_{m+1} - x_m}\right)^p (x_{m+1} - x_m)\right)^{\frac{1}{p}} \\
&= \left(\frac{2^p}{(x_{n+1} - x_n)^{p-1}} + \frac{2^p}{(x_{m+1} - x_m)^{p-1}}\right)^{\frac{1}{p}} \\
&\geq \left(2^p + 2^p\right)^{\frac{1}{p}} > 0.
\end{aligned}$$

Let

$$u_n(x) = f_n(x) + 1, \quad \forall\, n \geq 1.$$

It is obvious that u_n converges weakly to $u_0(x) \equiv 1$ and

$$\|u_n - u_m\| = \|f_n - f_m\| \geq (2^p + 2^p)^{\frac{1}{p}} > 0, \quad \forall\, n \geq 1. \tag{17}$$

Define a mapping $T : E \to E$ as follows

$$T_n(x) = \begin{cases} \frac{n}{n+1} u_n & \text{if } x = u_n (\exists\, n \geq 1), \\ -x & \text{if } x \neq u_n (\forall\, n \geq 1). \end{cases}$$

Since (17) holds, by using Conclusion 9 and 10, we know that $\{T_n\}$ is a uniformly closed family of countable quasi-nonexpansive mappings.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Competing interests
The authors declare that they have no competing interests.

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