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More results on Simpson's type inequality through convexity for twice differentiable continuous mappings

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Abstract

Our aim in this article is to incorporate the notion of "strongly s-convex function" and prove a new integral identity. Some new inequalities of Simpson type for strongly s-convex function utilizing integral identity and Holder's inequality are considered.

Keywords: Simpson's inequality, Strongly s-convex function, Integral identity, Holder's integral inequality

Mathematics Subject Classification: 26D15, 26E60, 41A55

Background

The following definition is well known in the literature as convex function:

Let $f : I \subseteq R \to R$ be a function defined on the interval I of real numbers. Then f is called convex, if $f\left(\lambda x + (1 - \lambda)y\right) \leq \lambda f(x) + (1 - \lambda)f\left(y\right)$, for all $x, y \in I$ and $\lambda \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on graph of f with Q between P and R, then Q is on or below chord PR.

Hudzik and Maligrada (1994) defined s-convex function as: A function $f : [0, \infty) \to R$ is said to be s-convex function in the first sense, if $f\left(\mu x + \nu y\right) \leq \mu^s f(x) + \nu^s f\left(y\right)$, for all $x, y \in [0, \infty)$ and $\mu, \nu \geq 0$, with $\mu^s + \nu^s = 1$. We denote this by K_α^1, for some fixed $s \in (0, 1]$. Also $f : [0, \infty) \to R$ is said to be s-convex function in the second sense, if above condition holds for all $x, y \in [0, \infty)$ and $\mu, \nu \geq 0$, with $\mu + \nu = 1$.

The following is very important and well known in the literature, as Simpson's inequality:

$$\left| \int_a^b f(x)dx - \frac{(b-a)}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^5, \quad (1)$$

where the mapping $f : [a, b] \to R$ is supposed to be a four times continuously differentiable on the interval (a, b) and having the fourth derivative bounded on (a, b), that is

$$\left\|f^{(4)}\right\|_\infty = \sup_{x \in (a,b)}\left|f^{(4)}(x)\right| < \infty.$$

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Dragomir et al. (2000) proved that: Let $f : [a, b] \to R$ be a differentiable function on I^0 (interior of I) $a, b \in I$ with $a < b$. If the mapping $|f'|$ is convex on $[a, b]$, then we have the following inequality:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \tag{2}$$

Sarikaya et al. (2010) showed that: Let $f : [a, b] \to R$ be a differentiable function on I^0 (interior of I) such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If the mapping $|f'|$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$, then we have the following inequality

$$\left| \frac{1}{6} \left[f(a) + 4f \left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right|$$
$$\leq \frac{(s-4)6^{s+1} + 2 \times 5^{s+2} - 2 \times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)} (b-a)(|f'(a)| + |f'(b)|). \tag{3}$$

Alomari et al. (2011) established that: Let $f : I \subset R \to R$ be a differentiable function on I^0 (interior of I) $a, b \in I$ with $a < b$. If the mapping $|f'|$ is s-convex on $[a, b]$, for some $s \in (0, 1]$, then we have the following inequality:

$$\left| f \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)dx \right|$$
$$\leq \frac{b-a}{4(s+1)(s+2)} \left\{ |f'(a)| + |f'(b)| + 2(s+1) \left| f' \left(\frac{a+b}{2} \right) \right| \right\} \tag{4}$$
$$\leq \frac{(2^{2-s} + 1)(b-a)}{4(s+1)(s+2)} \left[|f'(a)| + |f'(b)| \right].$$

For utilizing different kinds of convexities, additional findings relating to the Simpson's type inequality, readers are directed to Dragomir et al. (2000), Qaisar et al. (2013), Hussain and Qaisar (2014), Dragomir (1999), Wang et al. (2013), Xi and Qi (2013) and Pearce and Pecari'c (2000).

Main results

To prove our main result, we need the following definition and lemma.

Definition 1 (*Polyak* 1996) Let $f : I \subseteq R \to R$ is said to be strongly s-convex with modulus $c > 0$ and for some fixed $s \in (0, 1]$, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) - c\lambda(1-\lambda)(x-y)^2,$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Observation 2 It is clear that, any strongly s-convex function is a strong convex function but the converse is not true in general.

Now we prove the following lemma:

Lemma 3 *Let $f : I = [a,b] \subset R \to R$ be such that f' is absolutely continuous and $f'' \in L_1([a,b])$. Then the following inequality holds:*

$$\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+3b}{4}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{96}$$

$$\times \int_0^1 \psi(1-\psi)\left[f''\left(\frac{3+\psi}{4}a + \frac{1-\psi}{4}b\right) + f''\left(\frac{1+\psi}{4}a + \frac{3-\psi}{4}b\right) + f''\left(\frac{\psi}{4}a + \frac{4-\psi}{4}b\right) \right] d\psi.$$

(5)

Proof Using integrating by parts, we have

$$\int_0^1 \psi(1-\psi)f''\left(\frac{3+\psi}{4}a + \frac{1-\psi}{4}b\right)d\psi$$

$$= -\frac{4}{b-a}\left[\psi(1-\psi)f'\left(\frac{3+\psi}{4}a + \frac{1-\psi}{4}b\right)\Big|_0^1 - \int_0^1 (1-2\psi)f'\left(\frac{3+\psi}{4}a + \frac{1-\psi}{4}b\right)d\psi \right]$$

$$= -\frac{16}{(b-a)^2}\left[(1-2\psi)f\left(\frac{3+\psi}{4}a + \frac{1-\psi}{4}b\right)\Big|_0^1 + 2\int_0^1 f\left(\frac{3+\psi}{4}a + \frac{1-\psi}{4}b\right)d\psi \right]$$

$$= \frac{16}{(b-a)^2}\left[f(a) + f\left(\frac{3a+b}{4}\right) - \frac{96}{(b-a)^3}\int_a^{(3a+b)/4} f(x)dx \right].$$

Analogously,

$$\int_0^1 \psi(1-\psi)f''\left(\frac{1+\psi}{4}a + \frac{3-\psi}{4}b\right)d\psi$$

$$= \frac{16}{(b-a)^2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{96}{(b-a)^3}\int_{(3a+b)/4}^{(a+3b)/4} f(x)dx.$$

And

$$\int_0^1 \psi(1-\psi)f''\left(\frac{\psi}{4}a + \frac{4-\psi}{4}b\right)d\psi$$

$$= \frac{16}{(b-a)^2}\left[f\left(\frac{a+3b}{4}\right) + f(b) \right] - \frac{96}{(b-a)^3}\int_{(a+3b)/4}^{b} f(x)dx.$$

This proves as required.

Theorem 4 *Let $f : I = [a, b] \subset R \to R$ be such that f' is absolutely continuous and $f'' \in L_1([a, b])$. If the mapping $|f''|$ is strongly s-convex on $[a, b]$, for $q \geq 1$ and for some fixed $s \in (0, 1]$, then we have the following inequality:*

$$
\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4} \right) + 2f\left(\frac{a+3b}{4} \right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|
$$

$$
\leq \frac{6^{1/q}(b-a)^2}{576}\left\{
\begin{array}{l}
\left[\frac{(s-5)4^{s+2}+(s+9)3^{s+2}}{(s+1)(s+2)(s+3)4^s}|f''(a)|^q + \frac{1}{(s+2)(s+3)4^s}|f''(b)|^q \right.\\
\left. \quad - \frac{17c(b-a)^2}{960} \right]^{1/q}\\
+ \left[\frac{(s-1)2^{s+2}+s+5}{(s+1)(s+2)(s+3)4^s}|f''(a)|^q + \frac{(s-3)3^{s+2}+(s+7)2^{s+2}}{(s+1)(s+2)(s+3)4^s}|f''(b)|^q \right.\\
\left. \quad - \frac{37c(b-a)^2}{960} \right]^{1/q}\\
+ \left[\frac{1}{(s+2)(s+3)4^s}|f''(a)|^q + \frac{(s-5)4^{s+2}+(s+9)3^{s+2}}{(s+1)(s+2)(s+3)4^s}|f''(b)|^q - \frac{17c(b-a)^2}{960} \right]^{1/q}
\end{array}
\right\}.
$$

Proof Using Lemma 3 and strongly s-convexity of $|f''|^q$, we have

$$
\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4} \right) + 2f\left(\frac{a+3b}{4} \right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|
$$

$$
\leq \frac{(b-a)^2}{96}\left[
\begin{array}{l}
\int_0^1 \psi(1-\psi)\left|f''\left(\frac{3+\psi}{4}a + \frac{1-\psi}{4}b \right)\right|d\psi\\
+ \int_0^1 \psi(1-\psi)\left|f''\left(\frac{1+\psi}{4}a + \frac{3-\psi}{4}b \right)\right|d\psi + \int_0^1 \psi(1-\psi)\left|f''\left(\frac{\psi}{4}a + \frac{4-\psi}{4}b \right)\right|d\psi
\end{array}
\right]
$$

$$
\leq \frac{(b-a)^2}{96}\left[\int_0^1 \psi(1-\psi)d\psi \right]^{1-1/q}\left\{
\begin{array}{l}
\left[\int_0^1 \psi(1-\psi)\left(\begin{array}{l}\left(\frac{3+\psi}{4} \right)^s|f''(a)|^q + \left(\frac{1-\psi}{4} \right)^s|f''(b)|^q\\ - \frac{c(b-a)^2}{16}\int_0^1 \psi(1-\psi)^2(3+\psi)d\psi \end{array} \right) \right]^{1/q}\\
+ \left[\int_0^1 \psi(1-\psi)\left(\begin{array}{l}\left(\frac{1+\psi}{4} \right)^s|f''(a)|^q + \left(\frac{3-\psi}{4} \right)^s|f''(b)|^q\\ - \frac{c(b-a)^2}{16}\int_0^1 \psi(1-\psi^2)(3-\psi)d\psi \end{array} \right) \right]^{1/q}\\
+ \left[\int_0^1 \psi(1-\psi)\left(\begin{array}{l}\left(\frac{\psi}{4} \right)^s|f''(a)|^q + \left(\frac{4-\psi}{4} \right)^s|f''(b)|^q\\ - \frac{c(b-a)^2}{16}\int_0^1 \psi^2(1-\psi)(4-\psi)d\psi \end{array} \right) \right]^{1/q}
\end{array}
\right\}
$$

$$
= \frac{6^{1/q}(b-a)^2}{576}\left\{
\begin{array}{l}
\left[\frac{(s-5)4^{s+2}+(s+9)3^{s+2}}{(s+1)(s+2)(s+3)4^s}|f''(a)|^q + \frac{1}{(s+2)(s+3)4^s}|f''(b)|^q \right.\\
\left. \quad - \frac{17c(b-a)^2}{960} \right]^{1/q}\\
+ \left[\frac{(s-1)2^{s+2}+s+5}{(s+1)(s+2)(s+3)4^s}|f''(a)|^q + \frac{(s-3)3^{s+2}+(s+7)2^{s+2}}{(s+1)(s+2)(s+3)4^s}|f''(b)|^q \right.\\
\left. \quad - \frac{37c(b-a)^2}{960} \right]^{1/q}\\
+ \left[\frac{1}{(s+2)(s+3)4^s}|f''(a)|^q + \frac{(s-5)4^{s+2}+(s+9)3^{s+2}}{(s+1)(s+2)(s+3)4^s}|f''(b)|^q - \frac{17c(b-a)^2}{960} \right]^{1/q}
\end{array}
\right\}.
$$

Corollary 5 *Under the conditions of Theorem 4,*

1. If $q = 1$, then

$$\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+3b}{4}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^2}{96}\left[\frac{(s-3)4^{s+2} + (2s+6)\left(3^{s+2}+1\right)}{(s+1)(s+2)(s+3)4^s}\left(\left|f''(a)\right| + \left|f''(b)\right|\right) - \frac{71c(b-a)^2}{960} \right].$$

2. If $q = 1$ and $s = 1$, then

$$\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+3b}{4}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^2}{96}\left[\left(\left|f''(a)\right| + \left|f''(b)\right|\right) - \frac{71c(b-a)^2}{960} \right].$$

Theorem 6 *Let f be defined as in Theorem* 4 *and the mapping* $\left|f''\right|^q$ *is strongly s-convex on* $[a, b]$, *for* $q > 1$ *and for some fixed* $s \in (0, 1]$, *then we have the following inequality:*

$$\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+3b}{4}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^2}{96}\left[B\left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q}\left[\left(\frac{1}{4^s(s+1)}\right) \right]^{1/q}$$

$$\times \left\{ \begin{array}{l} \left[\left(4^{s+1} - 3^{s+1}\right)\left|f''(a)\right|^q + \left|f''(b)\right|^q - \frac{5c(b-a)^2(s+1)4^s}{48}\right]^{1/q} \\ + \left[\left(2^{s+1} - 1\right)\left|f''(a)\right|^q + \left(3^{s+1} - 2^{s+1}\right)\left|f''(b)\right|^q - \frac{11c(b-a)^2(s+1)4^s}{48}\right]^{1/q} \\ + \left[\left|f''(a)\right|^q + \left(4^{s+1} - 3^{s+1}\right)\left|f''(b)\right|^q - \frac{5c(b-a)^2(s+1)4^s}{48}\right]^{1/q} \end{array} \right\},$$

where $B(\alpha, \beta)$ is the classical Beta function which may be defined by

$$B(\alpha, \beta) = \int_0^1 \psi^{\alpha-1}(1-\psi)^{\beta-1}d\psi, \quad s, \beta > 0.$$

Proof Using Lemma 3, strong s-convexity of $\left|f''\right|^q$ and Holder's inequality, we have

$$\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4} \right) + 2f\left(\frac{a+3b}{4} \right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^2}{96}\left[\begin{array}{l} \int_0^1 \psi(1-\psi)\left| f''\left(\frac{3+\psi}{4}a + \frac{1-\psi}{4}b \right) \right|d\psi \\ + \int_0^1 \psi(1-\psi)\left| f''\left(\frac{1+\psi}{4}a + \frac{3-\psi}{4}b \right) \right|d\psi + \int_0^1 \psi(1-\psi)\left| f''\left(\frac{\psi}{4}a + \frac{4-\psi}{4}b \right) \right|d\psi \end{array} \right]$$

$$\leq \frac{(b-a)^2}{96}\left[\int_0^1 [\psi(1-\psi)]^{q/(q-1)}d\psi \right]^{1-1/q} \left\{ \begin{array}{l} \left[\left(\begin{array}{l} \int_0^1 \left(\left(\frac{3+\psi}{4} \right)^s |f''(a)|^q + \left(\frac{1-\psi}{4} \right)^s |f''(b)|^q \right) \\ - \frac{c(b-a)^2}{16}\int_0^1 (1-\psi)(3+\psi)d\psi \end{array} \right) \right]^{1/q} \\ + \left[\left(\begin{array}{l} \int_0^1 \left(\left(\frac{1+\psi}{4} \right)^s |f''(a)|^q + \left(\frac{3-\psi}{4} \right)^s |f''(b)|^q \right) \\ - \frac{c(b-a)^2}{16}\int_0^1 (1-\psi)(3-\psi)d\psi \end{array} \right) \right]^{1/q} \\ + \left[\left(\begin{array}{l} \int_0^1 \left(\left(\frac{\psi}{4} \right)^s |f''(a)|^q + \left(\frac{4-\psi}{4} \right)^s |f''(b)|^q \right) \\ - \frac{c(b-a)^2}{16}\int_0^1 \psi(4-\psi)d\psi \end{array} \right) \right]^{1/q} \end{array} \right\}$$

$$\times \left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4} \right) + 2f\left(\frac{a+3b}{4} \right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^2}{96}\left[B\left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q}\left[\left(\frac{1}{4^s(s+1)} \right) \right]^{1/q}$$

$$\times \left\{ \begin{array}{l} \left[\left(4^{s+1} - 3^{s+1} \right)|f''(a)|^q + |f''(b)|^q - \frac{5c(b-a)^2(s+1)4^s}{48} \right]^{1/q} \\ + \left[\left(2^{s+1} - 1 \right)|f''(a)|^q + \left(3^{s+1} - 2^{s+1} \right)|f''(b)|^q - \frac{11c(b-a)^2(s+1)4^s}{48} \right]^{1/q} \\ + \left[|f''(a)|^q + \left(4^{s+1} - 3^{s+1} \right)|f''(b)|^q - \frac{5c(b-a)^2(s+1)4^s}{48} \right]^{1/q} \end{array} \right\}.$$

This completes the proof.

Theorem 7 *Let f be defined as in Theorem* 4 *and the mapping* $|f''|^q$ *is strongly s-convex on* $[a,b],$ *for* $q > 1$ *and for some fixed* $s \in (0,1]$*, then we have the following inequality:*

$$\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4} \right) + 2f\left(\frac{a+3b}{4} \right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^2}{96}\left[\frac{(q-1)^2}{(2q-1)(3q-2)} \right]^{1-1/q}\left[\left(\frac{1}{4^s(s+1)(s+2)} \right) \right]^{1/q}$$

$$\times \left\{ \begin{array}{l} \left[\left((s-2)4^{s+1} + 3^{s+2} \right)|f''(a)|^q + |f''(b)|^q - \frac{7c(b-a)^2(s+1)(s+2)4^s}{192} \right]^{1/q} \\ + \left[\left(2^{s+1}s + 1 \right)|f''(a)|^q + \left(3^{s+2} - 2^{s+1}(s+4) \right)|f''(b)|^q - \frac{23c(b-a)^2(s+1)(s+2)4^s}{192} \right]^{1/q} \\ + \left[(s+1)|f''(a)|^q + \left(4^{s+2} - 3^{s+1}(s+5) \right)|f''(b)|^q - \frac{13c(b-a)^2(s+1)(s+2)4^s}{192} \right]^{1/q} \end{array} \right\}.$$

Proof　Using Lemma 3, Holder's inequality and strongly s- convexity of $\left|f''\right|^q$, we have

$$\left|\frac{1}{6}\left[f(a)+2f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+3b}{4}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{96}\left[\begin{array}{l}\int_0^1\psi(1-\psi)\left|f''\left(\frac{3+\psi}{4}a+\frac{1-\psi}{4}b\right)\right|d\psi\\+\int_0^1\psi(1-\psi)\left|f''\left(\frac{1+\psi}{4}a+\frac{3-\psi}{4}b\right)\right|d\psi+\int_0^1\psi(1-\psi)\left|f''\left(\frac{\psi}{4}a+\frac{4-\psi}{4}b\right)\right|d\psi\end{array}\right]$$

$$\leq \frac{(b-a)^2}{96}\left[\int_0^1[\psi(1-\psi)]^{q/(q-1)}d\psi\right]^{1-1/q}\left\{\begin{array}{l}\left[\left(\begin{array}{l}\int_0^1\psi\left(\left(\frac{3+\psi}{4}\right)^s\left|f''(a)\right|^q+\left(\frac{1-\psi}{4}\right)^s\left|f''(b)\right|^q\right)\\-\frac{c(b-a)^2}{16}\int_0^1\psi(1-\psi)(3+\psi)d\psi\end{array}\right)\right]^{1/q}\\+\left[\left(\begin{array}{l}\int_0^1\psi\left(\left(\frac{1+\psi}{4}\right)^s\left|f''(a)\right|^q+\left(\frac{3-\psi}{4}\right)^s\left|f''(b)\right|^q\right)\\-\frac{c(b-a)^2}{16}\int_0^1\psi(1+\psi)(3-\psi)d\psi\end{array}\right)\right]^{1/q}\\+\left[\left(\begin{array}{l}\int_0^1\psi\left(\left(\frac{\psi}{4}\right)^s\left|f''(a)\right|^q+\left(\frac{4-\psi}{4}\right)^s\left|f''(b)\right|^q\right)\\-\frac{c(b-a)^2}{16}\int_0^1\psi^2(4-\psi)d\psi\end{array}\right)\right]^{1/q}\end{array}\right\}.$$

or

$$\left|\frac{1}{6}\left[f(a)+2f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+3b}{4}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{96}\left[\frac{(q-1)^2}{(2q-1)(3q-2)}\right]^{1-1/q}\left[\left(\frac{1}{4^s(s+1)(s+2)}\right)\right]^{1/q}$$

$$\times\left\{\begin{array}{l}\left[((s-2)4^{s+1}+3^{s+2})\left|f''(a)\right|^q+\left|f''(b)\right|^q-\frac{7c(b-a)^2(s+1)(s+2)4^s}{192}\right]^{1/q}\\+\left[(2^{s+1}s+1)\left|f''(a)\right|^q+(3^{s+2}-2^{s+1}(s+4))\left|f''(b)\right|^q-\frac{23c(b-a)^2(s+1)(s+2)4^s}{192}\right]^{1/q}\\+\left[(s+1)\left|f''(a)\right|^q+(4^{s+2}-3^{s+1}(s+5))\left|f''(b)\right|^q-\frac{13c(b-a)^2(s+1)(s+2)4^s}{192}\right]^{1/q}\end{array}\right\}$$

This completes the proof.

Theorem 8　*Let f be defined as in Theorem 4 and the mapping $\left|f''\right|^q$ is strongly s-convex on $[a,b]$, for $q>1$ and for some fixed $s\in(0,1]$, then we have the following inequality:*

$$\left|\frac{1}{6}\left[f(a)+2f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+3b}{4}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{96}\left[\frac{(q-1)^2}{(2q-1)(3q-2)}\right]^{1-1/q}\left[\left(\frac{1}{4^s(s+1)(s+2)}\right)\right]^{1/q}$$

$$\times\left\{\begin{array}{l}\left[(4^{s+2}-(s+5)3^{s+1})\left|f''(a)\right|^q+(s+1)\left|f''(b)\right|^q-\frac{13c(b-a)^2(s+1)(s+2)4^s}{192}\right]^{1/q}\\+\left[(2^{s+2}-s-3)\left|f''(a)\right|^q+(3^{s+1}(s-1)-2^{s+2})\left|f''(b)\right|^q-\frac{21c(b-a)^2(s+1)(s+2)4^s}{192}\right]^{1/q}\\+\left[\left|f''(a)\right|^q+((s-2)4^{s+1}+3^{s+2})\left|f''(b)\right|^q-\frac{7c(b-a)^2(s+1)(s+2)4^s}{192}\right]^{1/q}\end{array}\right\}.$$

Proof   Using Lemma 3, Holder inequality and strongly s- convexity of $|f''|$, we have

$$\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4} \right) + 2f\left(\frac{a+3b}{4} \right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^2}{96}\left[\begin{array}{l} \int_0^1 \psi(1-\psi)\left| f''\left(\frac{3+\psi}{4}a + \frac{1-\psi}{4}b \right) \right| d\psi \\ + \int_0^1 \psi(1-\psi)\left| f''\left(\frac{1+\psi}{4}a + \frac{3-\psi}{4}b \right) \right| d\psi + \int_0^1 \psi(1-\psi)\left| f''\left(\frac{\psi}{4}a + \frac{4-\psi}{4}b \right) \right| d\psi \end{array} \right]$$

$$\leq \frac{(b-a)^2}{96}\left[\int_0^1 (1-\psi)\psi^{q/(q-1)}d\psi \right]^{1-1/q} \left\{ \begin{array}{l} \left[\left(\begin{array}{l} \int_0^1 (1-\psi)\left(\left(\frac{3+\psi}{4} \right)^s |f''(a)|^q + \left(\frac{1-\psi}{4} \right)^s |f''(b)|^q \right) \\ -\frac{c(b-a)^2}{16}\int_0^1 (1-\psi)^2(3+\psi)d\psi \end{array} \right) \right]^{1/q} \\ + \left[\left(\begin{array}{l} \int_0^1 (1-\psi)\left(\left(\frac{1+\psi}{4} \right)^s |f''(a)|^q + \left(\frac{3-\psi}{4} \right)^s |f''(b)|^q \right) \\ -\frac{c(b-a)^2}{16}\int_0^1 (1-\psi^2)(3-\psi)d\psi \end{array} \right) \right]^{1/q} \\ + \left[\left(\begin{array}{l} \int_0^1 (1-\psi)\left(\left(\frac{\psi}{4} \right)^s |f''(a)|^q + \left(\frac{4-\psi}{4} \right)^s |f''(b)|^q \right) \\ -\frac{c(b-a)^2}{16}\int_0^1 \psi(1-\psi)(4-\psi)d\psi \end{array} \right) \right]^{1/q} \end{array} \right\}.$$

or

$$\left| \frac{1}{6}\left[f(a) + 2f\left(\frac{3a+b}{4} \right) + 2f\left(\frac{a+3b}{4} \right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^2}{96}\left[\frac{(q-1)^2}{(2q-1)(3q-2)} \right]^{1-1/q} \left[\left(\frac{1}{4^s(s+1)(s+2)} \right) \right]^{1/q}$$

$$\times \left\{ \begin{array}{l} \left[\left(4^{s+2} - (s+5)3^{s+1} \right)|f''(a)|^q + (s+1)|f''(b)|^q - \frac{13c(b-a)^2(s+1)(s+2)4^s}{192} \right]^{1/q} \\ + \left[\left(2^{s+2} - s - 3 \right)|f''(a)|^q + \left(3^{s+1}(s-1) - 2^{s+2} \right)|f''(b)|^q - \frac{21c(b-a)^2(s+1)(s+2)4^s}{192} \right]^{1/q} \\ + \left[|f''(a)|^q + \left((s-2)4^{s+1} + 3^{s+2} \right)|f''(b)|^q - \frac{7c(b-a)^2(s+1)(s+2)4^s}{192} \right]^{1/q} \end{array} \right\}.$$

This completes the proof. $\square$

Conclusion

We incorporated notion of "strongly s-convex function" and proved a new integral identity. Some new inequalities of Simpson type for strongly s-convex function utilizing integral identity and Holder's inequality are also considered. These results give better estimates as presented earlier in the literature.

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