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On a conjecture of R. Brück and some linear differential equations

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Abstract

In this paper, we mainly investigate the Brück conjecture concerning entire function f and its differential polynomial $L_1(f) = a_k(z)f^{(k)} + \cdots + a_0(z)f$ sharing an entire function $\alpha(z)$ with $\sigma(\alpha) \leq \sigma(f)$, by using the theory of complex differential equation.

Keywords: Entire function, Brück conjecture, Difference polynomial

Mathematics Subject Classification: 39B32, 30D 35

Introduction and some results

It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna value distribution theory of meromorphic functions. For a meromorphic function f in the whole complex plane $\mathbb{C}$, we shall use the following standard notations of the value distribution theory:

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \ldots$$

(see Hayman 1964; Yang 1993; Yi and Yang 2003, 1995). We use $S(r, f)$ to denote any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to +\infty$, possibly outside of a set with finite measure. A meromorphic function $a(z)$ is called a small function with respect to f if $T(r,a) = S(r,f)$. In addition, we will use the notation $\sigma(f), \mu(f)$ to denote the order and the lower order of meromorphic function $f(z)$, which are defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r},$$

and

$$\mu(f) = \liminf_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$. We also use $\tau(f)$ to denote the type of an entire function $f(z)$ with $0 < \sigma(f) = \sigma < +\infty$, which is defined to be (see Hayman 1964)

$$\tau(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^{\sigma}}.$$

We use $\sigma_2(f)$ to denote the hyper-order of $f(z)$, which is defined to be (see Yi and Yang 2003, 1995)

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}.$$

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, for some $a \in \mathbb{C} \cup \{\infty\}$, if the zeros of $f(z) - a$ and $g(z) - a$ (if $a = \infty$, zeros of $f(z) - a$ and $g(z) - a$ are the poles of $f(z)$ and $g(z)$ respectively) coincide in locations and multiplicities we say that $f(z)$ and $g(z)$ share the value a *CM* (counting multiplicities) and if coincide in locations only we say that $f(z)$ and $g(z)$ share a *IM* (ignoring multiplicities).

Rubel and Yang (1977) proved the following result.

Theorem 1.1 Rubel and Yang (1977). *Let f be a nonconstant entire function. If f and f' share two finite distinct values CM, then $f \equiv f'$.*

In 1996, Brück proposed the following conjecture Brück (1996):

Conjecture 1.1 Brück (1996). *Let f be a non-constant entire function. Suppose that $\sigma_2(f)$ is not a positive integer or infinite, if f and f' share one finite value a CM, then*

$$\frac{f' - a}{f - a} = c$$

for some non-zero constant c.

Gundersen and Yang (1998) proved that Brück conjecture holds for entire functions of finite order and obtained the following result.

Theorem 1.2 [Gundersen and Yang (1998), Theorem 1]. *Let f be a nonconstant entire function of finite order. If f and f' share one finite value a CM, then $\frac{f'-a}{f-a} = c$ for some non-zero constant c.*

The shared value problems related to a meromorphic function f and its derivative $f^{(k)}$ have been a more widely studied subtopic of the uniqueness theory of entire and meromorphic functions in the field of complex analysis (see Chen et al. 2014; Li and Yi 2007; Liao 2015; Mues and Steinmetz 1986; Zhang and Yang 2009; Zhang 2005; Zhao 2012).

Li and Cao (2008) improved the Brück conjecture for entire function and its derivation sharing polynomials and obtained the following result:

Theorem 1.3 Li and Cao (2008). *Let Q_1 and Q_2 be two nonzero polynomials, and let P be a polynomial. If f is a nonconstant entire solution of the equation*

$$f^{(k)} - Q_1 = (f - Q_2)e^P,$$

then $\sigma_2(f) = \deg P$, where and in the following, $\deg P$ is the degree of P.

Mao (2009) studied the problem on Brück conjecture when $f^{(k)}$ is replaced by differential polynomial $L(f) = A_k f^{(k)} + \cdots + A_1 f' + A_0 f$ in Theorem 1.3.

Theorem 1.4 Mao (2009). *Let P(z) be a polynomial,* $A_k(z)(\not\equiv 0), \ldots, A_0(z)$ *be polynomials, and f be an entire function of order*

$$\sigma(f) > 1 + \max_{0 \le j \le k-1} \left\{ \frac{\deg A_j - \deg A_k}{k-j}, 0 \right\}$$

and hyper-order $\sigma_2(f) < \frac{1}{2}$. *If f and L(f) share P CM, then*

$$\frac{L(f) - P(z)}{f(z) - P(z)} = c,$$

for some constant $c \neq 0$, *where, and in the sequel,* $\deg A_j$ *denotes the degree of* $A_j(z)$, *k is a positive integer.*

Chang and Zhu (2009) further investigated the problem related to Brück conjecture and proved that Theorem 1.2 remains valid if the value a is replaced by a function $a(z)$.

Theorem 1.5 [Chang and Zhu (2009), Theorem 1] . *Let f be an entire function of finite order and a(z) be a function such that* $\sigma(a) < \sigma(f) < +\infty$. *If f and* f' *share a(z) CM, then* $\frac{f'-a}{f-a} = c$ *for some non-zero constant c.*

Thus, an interesting subject arises naturally about this problem: *what would happen when* $\sigma(a) < \sigma(f) < +\infty$ *is replaced by* $0 < \sigma(a) = \sigma(f) < +\infty$ *in Theorems 1.2–1.5?*

Conclusions

Motivated by the above question, the main purpose of this article is to study the growth of solution of differential equation on entire function f and its linear differential polynomial

$$L_1(f) = a_k(z)f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f, \tag{1}$$

where k is a positive integer, $a_k(z)(\not\equiv 0), a_{k-1}(z), \ldots, a_1(z)$ and $a_0(z)$ are polynomials, and obtain the following theorems.

Theorem 2.1 *Let f(z) and* $\alpha(z)$ *be two nonconstant entire functions and satisfy* $0 < \sigma(\alpha) = \sigma(f) < +\infty$ *and* $\tau(f) > \tau(\alpha)$, *and let P(z) be a polynomial such that*

$$\sigma(f) > \deg P + \max \left\{ \frac{deg a_j - deg a_k}{k-j}, 0 \right\}. \tag{2}$$

If f is a nonconstant entire solution of the following differential equation

$$L_1(f) - \alpha(z) = (f(z) - \alpha(z))e^{P(z)}, \tag{3}$$

where $L_1(f)$ *is stated as in (1). Then P(z) is a constant.*

If $L_1(f)$ *is replaced by the following linear differential polynomial* $L_2(f)$

$$L_2(f) = a_k(z)f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f + \beta(z), \tag{4}$$

where k is a positive integer, $a_k(z)(\not\equiv 0), a_{k-1}(z), \ldots, a_1(z)$ and $a_0(z)$ are polynomials, and β is an entire function satisfying either $\sigma(\beta) < \mu(f)$ or $0 < \sigma(\beta) = \sigma(f) < +\infty$ and $\tau(\beta) < \tau(f)$, then we obtain the following results.

Theorem 2.2 *Let f(z) and $\alpha(z)$ be two nonconstant entire functions and satisfy $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\alpha)$, and let P(z) be a polynomial satisfying (2). If f is a nonconstant entire solution of the following differential equation*

$$L_2(f) - \alpha(z) = (f(z) - \alpha(z))e^{P(z)}, \tag{5}$$

where $L_2(f)$ is stated as in (4) and β is an entire function satisfying $0 < \sigma(\beta) = \sigma(f) < +\infty$ and $\tau(\beta) < \tau(f)$. Then P(z) is a constant.

Theorem 2.3 *Let f(z) and $\alpha(z)$ be two nonconstant entire functions and satisfy $\sigma(\alpha) < \mu(f)$, and let P(z) be a polynomial satisfying (2). If f is a nonconstant entire solution of Eq. (5), where $L_2(f)$ is stated as in (4) and β is an entire function satisfying $\sigma(\beta) < \mu(f)$. Then $\sigma_2(f) = \deg P$.*

Corollary 2.1 *Let f(z) and $\alpha(z)$ be two nonconstant entire functions and satisfy $\sigma(\alpha) < \mu(f)$, and let P(z be a polynomial satisfying) (2). If f is a nonconstant entire solution of Eq. (3),where $L_1(f)$* is stated as in (1). *Then $\sigma_2(f) = \deg P$.*

Some Lemmas

To prove our theorems, we will require some lemmas as follows.

Lemma 3.1 Laine (1993). *Let f(z) be a transcendental entire function, $\nu(r,f)$ be the central index of f(z). Then there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure, we choose z satisfying $|z| = r \notin [0, 1] \cup E$ and $|f(z)| = M(r,f)$, we get*

$$\frac{f^{(j)}(z)}{f(z)} = \left\{\frac{\nu(r,f)}{z}\right\}^j (1 + o(1)), \ \textit{for} \ j \in N.$$

Lemma 3.2 He and Xiao (1988). *Let f(z) be an entire function of finite order $\sigma(f) = \sigma < +\infty$, and let $\nu(r,f)$ be the central index of f. Then*

$$\limsup_{r\to+\infty} \frac{\log \nu(r,f)}{\log r} = \sigma(f).$$

And if f is a transcendental entire function of hyper order $\sigma_2(f)$, then

$$\limsup_{r\to+\infty} \frac{\log\log \nu(r,f)}{\log r} = \sigma_2(f).$$

Lemma 3.3 Mao (2009). *Let f be a transcendental entire function and let $E \subset [1, +\infty)$ be a set having finite logarithmic measure. Then there exists $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n,f), \theta_n \in [0, 2\pi)$, $\lim_{n\to+\infty} \theta_n = \theta_0 \in [0, 2\pi), r_n \notin E$ and if $0 < \sigma(f) < +\infty$, then for any given $\varepsilon > o$ and sufficiently large r_n,*

$$r_n^{\sigma(f)-\varepsilon} < \nu(r_n,f) < r_n^{\sigma(f)+\varepsilon}.$$

If $\sigma(f) = +\infty$, then for any given large $M > 0$ and sufficiently large r_n, $\nu(r_n,f) > r_n^M$.

Lemma 3.4 Laine (1993). *Let* $P(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$ *with* $b_n \neq 0$ *be a polynomial. Then, for every* $\varepsilon > 0$, *there exists* $r_0 > 0$ *such that for all* $r = |z| > r_0$ *the inequalities*

$$(1-\varepsilon)|b_n|r^n \leq |P(z)| \leq (1+\varepsilon)|b_n|r^n$$

hold.

Lemma 3.5 *Let* $f(z)$ *and* $A(z)$ *be two entire functions with* $0 < \sigma(f) = \sigma(A) = \sigma < +\infty, 0 < \tau(A) < \tau(f) < +\infty$, *then there exists a set* $E \subset [1, +\infty)$ *that has infinite logarithmic measure such that for all* $r \in E$ *and a positive number* $\kappa > 0$, *we have*

$$\frac{M(r,A)}{M(r,f)} < \exp\{-\kappa r^{\sigma}\}.$$

Proof By definition, there exists an increasing sequence $\{r_m\} \to +\infty$ satisfying $(1 + \frac{1}{m})r_m < r_{m+1}$ and

$$\lim_{m \to +\infty} \frac{\log M(r_m, f)}{r_m^{\sigma}} = \tau(f). \tag{6}$$

For any given $\beta(\tau(A) < \beta < \tau(f))$, then there exists some positive integer m_0 such that for all $m \geq m_0$ and for any given $\varepsilon(0 < \varepsilon < \tau(f) - \beta)$, we have

$$\log M(r_m, f) > (\tau(f) - \varepsilon)r_m^{\sigma}. \tag{7}$$

Thus, there exists some positive integer m_1 such that for all $m \geq m_1$, we have

$$\left(\frac{m}{m+1}\right)^{\sigma} > \frac{\beta}{\tau(f) - \varepsilon}. \tag{8}$$

From (6–8), for all $m \geq m_2 = \max\{m_0, m_1\}$ and for any $r \in [r_m, (1 + \frac{1}{m})r_m]$, we have

$$\begin{aligned} M(r,f) &\geq M(r_m, f) > \exp\{(\tau(f) - \varepsilon)r_m^{\sigma}\} \\ &\geq \exp\left\{(\tau(f) - \varepsilon)\left(\frac{m}{m+1} r\right)^{\sigma}\right\} > \exp\{\beta r^{\sigma}\}. \end{aligned} \tag{9}$$

Set $E = \bigcup_{m=m_2}^{\infty} [r_m, (1 + \frac{1}{m})r_m]$, then

$$m_l E = \sum_{m=m_2}^{\infty} \int_{r_m}^{(1+\frac{1}{m})r_m} \frac{dt}{t} = \sum_{m=m_2}^{\infty} \log\left(1 + \frac{1}{m}\right) = \infty.$$

From the definition of type of entire function, for any sufficiently small $\varepsilon > 0$, we have

$$M(r,A) < \exp\{\tau(A) + \varepsilon)r^{\sigma}\}. \tag{10}$$

By (9) and (10), set $\kappa = \beta - \tau(A) - \varepsilon$, for all $r \in E$, we have

$$\frac{M(r,A)}{M(r,f)} < \exp\{-(\beta - \tau(A) - \varepsilon)r^{\sigma}\} = e^{-\kappa r^{\sigma}}.$$

Thus, this completes the proof of this lemma. □

The proof of Theorem 2.1

Proof Since $P(z)$ is a polynomial, assume that $\deg P = m \geq 1$. Let

$$P(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0,$$

where $b_m, \ldots, b_0$ are constants and $b_m \neq 0, m \geq 1$. Thus, it follows from (3) and Lemma 3.4 that

$$|b_m| r^m (1 + o(1)) = |P(z)| = \left| \log \frac{\frac{L_1(f(z))}{f(z)} - \frac{\alpha(z)}{f(z)}}{1 - \frac{\alpha(z)}{f(z)}} \right|. \tag{11}$$

Since $L_1(f) = a_k f^k + a_{k-1} f^{(k-1)} + \cdots + a_0 f$, from Lemma 3.1, then there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure, such that for some point $|z| = re^{i\theta} (\theta \in [0, 2\pi)), r \notin E_1$ and $M(r,f) = |f(z)|$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left\{ \frac{\nu(r,f)}{z} \right\}^j (1 + o(1)), \quad 1 \leq j \leq k.$$

Thus, it follows that

$$\begin{aligned} \frac{L_1(f(z))}{f(z)} &= a_k \left\{ \frac{\nu(r,f)}{z} \right\}^k (1 + o(1)) + \cdots + a_1 \left\{ \frac{\nu(r,f)}{z} \right\} (1 + o(1)) + a_0 \\ &= \frac{a_k}{z^k} (1 + o(1)) \left[\nu(r,f)^k + \sum_{j=1}^{k} \frac{a_{k-j}}{a_k} z^j \nu(r,f)^{k-j} (1 + o(1)) \right]. \end{aligned} \tag{12}$$

From Lemma 3.3, there exists $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n,f), \theta_n \in [0, 2\pi)$, $\lim_{n\to\infty} \theta_n = \theta_0 \in [0, 2\pi), r_n \notin E_1$, then for any given ε satisfying

$$0 < \varepsilon < \min_{1 \leq j \leq k} \frac{j[\sigma(f) - \deg P - \frac{d_{k-j}}{j}]}{3k - j},$$

where $d_{k-j} = deg a_{k-j} - deg a_k$, and sufficiently large r_n, we have

$$r_n^{\sigma(f)-\varepsilon} < \nu(r_n,f) < r_n^{\sigma(f)+\varepsilon}. \tag{13}$$

Since $a_0(z), \ldots, a_k(z)$ are polynomials, let $a_j(z) = \sum_{t=0}^{s_j} l_{jt} z^t$, where $s_j = \deg a_j, j = 0, 1, \ldots, k$. Then, from Lemma 3.4 and (13), we have

$$\left|\frac{a_{k-j}}{a_k}z^j\nu(r,f)^{k-j}(1+o(1))\right| \leq M\frac{|l_{k-j,s_{k-j}}|r_n^{s_{k-j}}}{|l_{k,s_k}|r_n^{s_k}}r_n^j r_n^{(\sigma(f)+\varepsilon)(k-j)}$$
$$= M\frac{|l_{k-j,s_{k-j}}|}{|l_{k,s_k}|}r_n^{d_{k-j}+j+(\sigma(f)+\varepsilon)(k-j)}$$
$$\leq M\frac{|l_{k-j,s_{k-j}}|}{|l_{k,s_k}|}r_n^{k\sigma(f)-j\sigma(f)+d_{k-j}+\deg P+(k-j)\varepsilon}, \tag{14}$$

where $d_{k-j} = s_{k-j} - s_k$ and M is a positive constant. Since $-j\sigma(f) + d_{k-j} + \deg P + (k-j)\varepsilon < -2k\varepsilon < 0$, it follows from (14) that

$$\left|\frac{a_{k-j}}{a_k}z^j\nu(r_n,f)^{k-j}(1+o(1))\right| < M\frac{|l_{k-j,s_{k-j}}|}{|l_{k,s_k}|}r_n^{k(\sigma(f)-2\varepsilon)}$$
$$= o(\nu(r_n,f)^k), \quad as\ r_n \to +\infty, \quad r_n \notin E_1. \tag{15}$$

Since $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(\alpha) < \tau(f) < +\infty$, from Lemma 3.5, there exists a set $E \subset [1,+\infty)$ that has infinite logarithmic measure such that for a sequence $\{r_n\}_1^\infty \in E_2 = E - E_1$, we have

$$\frac{M(r,\alpha)}{M(r,f)} < \exp\{-\kappa r_n^{\sigma(f)}\} \to 0, \quad as\ n \to +\infty. \tag{16}$$

From (11), (12), (15), (16) and Lemma 3.2, we can get that

$$|b_m|r_n^m(1+o(1)) = |P(z)| = O(\log r_n), \tag{17}$$

which is impossible. Thus, $P(z)$ is not a polynomial, that is, $P(z)$ is a constant.

Thus, this completes the proof of Theorem 2.1. □

The proof of Theorem 2.2

Proof First of all, we rewrite (5) as

$$\frac{L_2(f)-\alpha(z)}{f(z)-\alpha(z)} = \frac{\frac{L_1(f)}{f}+\frac{\beta(z)}{f(z)}-\frac{\alpha(z)}{f(z)}}{1-\frac{\alpha(z)}{f(z)}} = e^{P(z)}, \tag{18}$$

where $L_1(f)$ is stated as in Theorem 2.1. Since $0 < \sigma(f) = \sigma(\alpha) = \sigma(\beta) < +\infty$, $\tau(\alpha) < \tau(f)$ and $\tau(\beta) < \tau(f)$, from Lemma 3.5, there exists a set $E \subset [1,+\infty)$ that has infinite logarithmic measure such that for a sequence $\{r_n\}_1^\infty \in E_3 = E - E_1$, we have

$$\frac{M(r,\alpha)}{M(r,f)} < \exp\{-\kappa r_n^{\sigma(f)}\} \to 0, \ and\ \frac{M(r,\beta)}{M(r,f)} < \exp\{-\kappa r_n^{\sigma(f)}\} \to 0, \quad as\ n \to +\infty.$$

Then by using the proceeding as in proof of Theorem 2.1, we prove that $P(z)$ is a constant.

This completes the proof of Theorem 2.2. □

The proof of Theorem 2.3.

Proof From $P(z)$ is a polynomial, we will consider two cases (i) $\sigma(f) < +\infty$ and (ii) $\sigma(f) = +\infty$.

Case 1. Suppose that $\sigma(f) < +\infty$. Then $\sigma_2(f) = 0$. Since $\sigma(\alpha) < \mu(f), \sigma(\beta) < \mu(f)$, from Definitions of the order and the lower order, there exists infinite sequence $\{z_n\}_1^{\infty}$, we have

$$\frac{|\alpha(z_n)|}{|f(z_n)|} \to 0, \ \ and \ \ \frac{|\beta(z_n)|}{|f(z_n)|} \to 0, \quad as \ \ n \to \infty.$$

Thus, by using the same argument as in Theorem 2.1, we can get that $P(z)$ is a constant, that is, $\deg P = 0$. Therefore, $\sigma_2(f) = \deg P$.

Case 2. Suppose that $\sigma(f) = +\infty$. Set $F(z) = f(z) - \alpha(z)$. Since $\sigma(\alpha) < \mu(f)$, it follows from (2) that

$$\sigma(F) = +\infty, \ \ \sigma_2(F) = \sigma_2(f), \tag{19}$$

and

$$\sigma(F) > \deg P + \max\left\{\frac{dega_j - dega_k}{k-j}, 0\right\}. \tag{20}$$

Furthermore, we can rewrite (4) as

$$a_k(z)\frac{F^{(k)}(z)}{F(z)} + \cdots + a_1(z)\frac{F'(z)}{F(z)} + a_0(z) + \frac{\gamma(z)}{F(z)} = e^{P(z)}, \tag{21}$$

where $\gamma(z) = a_k\alpha^{(k)} + \cdots + a_1\alpha + \beta - \alpha$. Since $\sigma(\beta) < \mu(f)$, $\sigma(\alpha) < \mu(f)$ and $a_i(z), (i = 0, \ldots, k)$ are polynomials, we have

$$\sigma(\gamma) \leq \max\{\sigma(\alpha), \sigma(\beta)\} < \mu(f) \leq \sigma(f). \tag{22}$$

From Lemma 3.1, there exists a set $E_4 \subset (1, +\infty)$ with finite logarithmic measure, we choose z satisfying $|z| = r \notin [0,1] \cup E_4$ and $|F(z)| = M(r, F)$, we get

$$\frac{F^{(j)}(z)}{F(z)} = \left\{\frac{\nu(r,F)}{z}\right\}^j (1 + o(1)), \ \ for \ \ j \in 1, 2, \ldots, k. \tag{23}$$

Since $\sigma(F) = +\infty$, then it follows from Lemma 3.3 that there exists $\{z_n = r_n e^{i\theta_n}\}$ with $|F(z_n)| = M(r_n, F), \theta_n \in [0, 2\pi)$, $\lim_{n\to\infty}\theta_n = \theta_0 \in [0, 2\pi), r_n \notin E_5$, such that for any large constant K and for sufficiently large r_n we have

$$\nu(r_n, F) \geq r_n^K. \tag{24}$$

From $M(r_n, F) = |F(z_n)|$, $F(z), \gamma(z)$ are entire functions and (18), by using definitions of the order and the lower order, we have

$$\left|\frac{\gamma(z_n)}{F(z_n)}\right| \to 0, \quad as \ \ r \to +\infty. \tag{25}$$

Thus, it follows from (21), (23)–(25) that

$$a_k\left(\frac{\nu(r_n,F)}{z_n}\right)^k(1+o(1))=e^{P(z_n)}. \tag{26}$$

Let

$$P(z)=b_mz^m+b_{m-1}z^{m-1}+\cdots+b_0,$$

where $b_m,\ldots,b_0$ are constants and $b_m\neq 0, m\geq 1$. From Lemma 3.4, there exists sufficiently large positive number r_0 and $n_0\in\mathbb{N}_+$, such that for sufficiently large positive integer $n>n_0$ satisfying $|z_n|=r_n>r_0$, we have for every $\varepsilon'>0$

$$\log|b_m|+m\log|z_n|+\log|1-\varepsilon'|\leq\log|P(z_n)|\leq|\log\log e^{P(z_n)}|. \tag{27}$$

It follows from (26) that

$$\begin{aligned}|\log\log e^{P(z_n)}|&\leq\log|\log|a_k||+\log\log\nu(r_n,F)+\log\log r_n+O(1)\\&\leq\log\log\nu(r_n,F)+O(\log\log r_n).\end{aligned} \tag{28}$$

Thus, we have from (27), (28) and Lemma 3.2 that

$$m=\deg P(z)\leq\sigma_2(F)=\sigma_2(f). \tag{29}$$

On the other hand, since a_k is a polynomial, it follows from (27) and Lemma 3.4 that

$$M(r_n,e^{P(z_n)})\geq K_1r_n^{d_k}\left(\frac{\nu(r_n,F)}{r_n}\right)^k,$$

where $K_1>0$ is a constant. Then we have

$$\nu(r_n,F)^k\leq K_1^{-1}r_n^{k-d_k}M(r_n,e^{P(z_n)}). \tag{30}$$

Thus, it follows from (30) and Lemma 3.2 that

$$\begin{aligned}\sigma_2(f)=\sigma_2(F)&=\limsup_{r_n\to+\infty}\frac{\log\log\nu(r_n,F)}{\log r_n}=\limsup_{r_n\to+\infty}\frac{\log\log\nu(r_n,F)^k}{\log r_n}\\&\leq\limsup_{r_n\to+\infty}\frac{\log\log K_1^{-1}r_n^{k-d_k}M(r_n,e^{P(z_n)})}{\log r_n}=\sigma(e^P).\end{aligned}$$

Since $P(z)$ is a polynomial, then $\sigma(e^P)=\deg P=m$. By combining (29), we have $\sigma_2(f)=\deg P$.

Therefore, this completes the proof of Theorem 2.3. □

Authors' contributions
HYX and LZY completed the main part of this article. Both authors read and approved the final manuscript.

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Competing interests
The authors declare that they have no competing interests.

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