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Krasnoselskii-type algorithm for zeros of strongly monotone Lipschitz maps in classical banach spaces

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Abstract

Let $E = L_p, 1 < p < \infty$, and $A : E \to E^*$ be a strongly monotone and Lipschitz mapping. A Krasnoselskii-type sequence is constructed and proved to converge strongly to the unique solution of $Au = 0$. Furthermore, our technique of proof is of independent interest.

Keywords: Strongly monotone, Lipschitz, Hölder continiuty

Background

Let H be a real Hilbert space. An operator $A : H \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \forall\, x, y \in H, \tag{1.1}$$

and is called *strongly monotone* if there exists $\lambda \in (0, 1)$ such that

$$\langle Ax - Ay, x - y \rangle \geq \lambda \|x - y\|^2 \quad \forall x, y \in H. \tag{1.2}$$

Interest in monotone operators stems mainly from their usefulness in numerous applications. Consider, for example, the following: Let $f : H \to \mathbb{R}$ be a proper and convex function. The *subdifferential* of f at $x \in H$ is defined by

$$\partial f(x) = \left\{ x^* \in H : f(y) - f(x) \geq \langle y - x, x^* \rangle \quad \forall\, y \in H \right\}.$$

It is easy to check that $\partial f : H \to 2^H$ is a *monotone operator* on H, and that $0 \in \partial f(x)$ if and only if x is a minimizer of f. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of f.

Let E be a real normed space, E^* its topological dual space. The map $J : E \to 2^{E^*}$ defined by

$$Jx = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|.\|x^*\|,\ \|x\| = \|x^*\| \right\}$$

is called the *normalized duality map* on E. A map $A : E \to E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0. \tag{1.3}$$

A is called *strongly accretive* if there exists $k \in (0, 1)$ such that for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq k\|x - y\|^2. \tag{1.4}$$

Several existence theorems have been established for the equation $Au = 0$ when A is of the monotone-type (see e.g., Deimling (1985; Pascali and Sburian 1978).

For approximating a solution of $Au = 0$, assuming existence, where $A : E \to E$ is of accretive-type, Browder (1967) defined an operator $T : E \to E$ by $T := I - A$, where I is the identity map on E. He called such an operator *pseudo-contractive*. A map $T : E \to E$ is then called *pseudo-contractive* if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 \quad \forall x, y \in E, \tag{1.5}$$

and is called *strongly pseudo-contractive* if there exists $k \in (0, 1)$ such that

$$\langle Tx - Ty, x - y \rangle \leq k\|x - y\|^2 \quad \forall x, y \in E. \tag{1.6}$$

It is trivial to observe that zeros of A corresspond to fixed points of T. For Lipschitz strongly pseudo-contractive maps, Chidume (1987) proved the following theorem.

Theorem C1 (Chidume 1987) *Let $E = L_p$, $2 \leq p < \infty$, and $K \subset E$ be nonempty closed convex and bounded. Let $T : K \to K$ be a strongly pseudocontractive and Lipschitz map. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$, $n \geq 0$, where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T.*

The main tool used in the proof of Theorem C1 is an inequality of Bynum (1976). This theorem signalled the return to extensive research efforts on inequalities in Banach spaces and their applications to iterative methods for solutions of nonlinear equations. Consequently, this theorem of Chidume has been generalized and extended in various directions, leading to flourishing areas of research, for the past thirty years or so, by numerous authors (see e.g., Chidume 1986, 1990, 2002; Chidume and Ali 2007; Chidume and Chidume 2005, 2006; Chidume and Osilike 1999; Deng 1993a, b; Zhou 1997; Zhou and Jia 1996, 1997; Liu 1995, 1997; Qihou 1990, 2002; Weng 1991, 1992; Xiao 1998; Xu 1989, 1991a, b, 1992, 1998; Xu and Roach 1991, 1992; Xu et al. 1995; Zhu 1994 and a host of other authors). Recent monographs emanating from these researches include those by Chidume (2009), Berinde (2007), Goebel and Reich (1984) and William and Shahzad (2014).

Unfortunately, the success achieved in using geometric properties developed from the mid 1980ies to early 1990ies in approximating zeros of accretive-type mappings has not carried over to approximating zeros of monotone-type operators in general Banach spaces. The first problem is that since A maps E to E^*, for $x_n \in E$, Ax_n is in E^*. Consequently, a recursion formula containing x_n and Ax_n may not be well defined. Another difficulty is that the normalized duality map which appears in most Banach space inequalities developed, and also appears in the definition of accretive-type mappings, does not appear in the definition of monotone-type mappings in general Banach spaces. This creats very serious technical difficulties.

Attemps have been made to overcome the first difficulty by introducing the inverse of the normalized duality mapping in the recursion formulas for approximating zeros of monotone-type mappings. But one major problem with such recursion formulas is that the exact form of the normalized duality map (or its inverse) is not known precisely in any space more general than L_p spaces, $1 < p < \infty$. Futhermore, the recursion formulas, apart from containing the normalized duality map and its inverse, generally involve computation of subsets and generalized projections, both of which are defined in a way that makes their computation almost impossible. We give some examples of some results obtained using these approximation schemes. Before we do this, however, we need the following definitions.

Let E be a real normed space and let a funtion $\phi(.,.) : X \times X \longrightarrow \mathbb{R}$ be defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, J(y) \rangle + ||y||^2 \quad \forall\, x, y \in E.$$

It is easy to see that in Hilbert space, $\phi(x, y)$ reduces to $\|x - y\|^2$. A function $\pi_K : E \longrightarrow K$ defined by: $\pi_K(x) = \bar{x}$ such that $\bar{x}$ is the solution of

$$\min\left\{\phi(x, y), y \in K\right\},$$

is called a *generalized projection map.*

Now we present the following results.

In Hilbert space, suppose that a map $A : K \to H$ is γ-inverse strongly monotone, i.e., there exists $\gamma > 0$ such that $\langle Ax - Ay, x - y \rangle \geq \gamma ||Ax - Ay||^2 \,\forall\, x, y \in H$. Iiduka et al. (2004) studied the following iterative scheme.

$$
\begin{cases}
x_0 & \in K, choosen\ arbitrary, \\
y_n & = P_K(x_n - \alpha_n A x_n); \\
C_n & = \left\{z \in K : \|y_n - z\| \leq \|x_n - z\|\right\}, \\
Q_n & = \left\{z \in K : \langle x_n - z, x_0 - x_n \rangle \geq 0\right\} \\
x_{n+1} & = P_{C_n \cap Q_n}(x_0), n \geq 1,
\end{cases}
\tag{1.7}
$$

where $\{\alpha_n\}$ is a sequence in $[0, 2\gamma]$. They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $P_{VI(K,A)}(x_0)$, where $P_{VI(K,A)}$ is the metric projection from K onto $VI(K, A)$ (see e.g., Iiduka et al. 2004 for definition and explanation of the symbols).

Zegeye and Shahzad proved the following result.

Theorem 1.1 (Zegeye and Shahzad 2009) *Let E be uniformly smooth and 2-uniformly convex real Banach space with dual E^*. Let*

$A : E \longrightarrow E^*$ be a γ-inverse strongly monotone mapping and

$T : E \longrightarrow E$ be relatively weak nonexpansive mapping with $A^{-1}(0) \cap F(T) \neq \emptyset$. Assume that $0 < \alpha_n \leq b_0 := \frac{\gamma c^2}{2}$, where c is the constants from the Lipschitz property of J^{-1}, then the sequence generated by

$$
\begin{cases}
x_0 & \in K, choosen\ arbitrary, \\
y_n & = J^{-1}(Jx_n - \alpha_n A x_n); \\
z_n & = Ty_n, \\
H_0 & = \left\{v \in K : \phi(v, z_0) \leq \phi(v, y_0) \leq \phi(v, x_0)\right\}, \\
H_n & = \left\{v \in H_{n-1} \cap W_{n-1} : \phi(v, z_n) \leq \phi(v, y_n) \leq \phi(v, x_n)\right\}, \\
W_0 & = E, \\
W_n & = \left\{v \in W_{n-1} \cap H_{n-1} : \langle x_n - v, jx_0 - jx_n \rangle \geq 0\right\} \\
x_{n+1} & = \Pi_{H_n \cap W_n}(x_0), n \geq 1,
\end{cases}
$$

converges strongly to $\Pi_{F(T) \cap A^{-1}(0)} x_0$ *where* $\Pi_{F(T) \cap A^{-1}(0)}$ *is the generalised projection from E onto* $F(T) \cap A^{-1}(0)$.

We remark here that although the approximation methods used in the result of Iiduka *et al.* referred to above, and in Theorem 1.1 yield strong convergence to a solution of the problem under consideration, it is clear that they are not easy to implement. Furthermore, Theorem 1.1 excludes L_p spaces, $2 < p < \infty$, because these spaces are *not* 2-uniformly convex. The theorem, however, is applicable in L_p spaces $1 < p < 2$.

In this paper, we introduce an *iterative scheme of Krasnoselskii-type* to approximate the unique zero of *a strongly monotone Lipschitz mapping* in L_p spaces, $1 < p < \infty$. In these spaces, the formula for J is known precisely (see e.g., Cioranescu 1990; Chidume 2009). The Krasnoselskii sequence, whenever it converges, is known to converge as fast as a geometric progression. Furthermore, our iteration method which will not involve construction of subsets or the use of generalized projection is also of independent interest.

Preliminaries

In the sequel, we shall need the following results and definitions.

Lemma 2.1 (see e.g., Chidume 2009, p. 55) *Let $E = L_p$, $1 < p < 2$. Then, there exists a constant $c_p > 0$ such that for all x, y in L_p the following inequalities hold:*

$$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, J(x) \rangle + c_p \|y\|^2, \tag{2.1}$$

$$\langle x - y, J(x) - J(y) \rangle \geq (p - 1) \|x - y\|^2. \tag{2.2}$$

Let E be a smooth real Banach space with dual E^*. The function $\phi : E \times E \to \mathbb{R}$, defined by,

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for } x, y \in E, \tag{2.3}$$

where J is the normalized duality mapping from E into 2^{E^*}, introduced by Alber has been studied by Alber (1996), Alber and Guerre-Delabriere (2001), Kamimura and Takahashi (2002), Reich (1996) and a host of other authors. If $E = H$, a real Hilbert space, then Eq (2.3) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \text{for } x, y \in E. \tag{2.4}$$

Define $V : X \times X^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2. \tag{2.5}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)) \quad \forall\, x \in X, \, x^* \in X^*. \tag{2.6}$$

Corollary 2.2 *Let $E = L_p, 1 < p \leq 2$. Then J^{-1} is Lipschitz, i.e., there exists $L_1 > 0$ such that for all $u, v \in E^*$, the following inequality holds:*

$$\|J^{-1}(u) - J^{-1}(v)\| \leq L_1 \|u - v\|. \tag{2.7}$$

Proof This follows from inequality (2.2). □

For L_p, $2 \le p < \infty$, we have the following lemma.

Lemma 2.3 (Alber and Ryazantseva 2006, p. 48) *Let* $X = L_p$, $p \ge 2$. *Then, the inverse of the normalized duality map* $J^{-1} : X^* \to X$ *is Hölder continuous on balls. i.e.,* $\forall u, v \in X^*$ *such that* $\|u\| \le R$ *and* $\|v\| \le R$, *then*

$$\|J^{-1}(u) - J^{-1}(v)\| \le m_p \|u - v\|^{\frac{1}{p-1}}, \tag{2.8}$$

where $m_p := (2^{p+1}Lpc_2^p)^{\frac{1}{p-1}} > 0$, *for some constant* $c_2 > 0$.

Proof This follows from the following inequality:

$$\langle Jx - Jy, x - y \rangle \ge \frac{\|x - y\|^p}{2^{p+1}Lpc_2^p}, \quad c_2 = 2\max\{1, R\}. \tag{2.9}$$

(see e.g., Alber and Ryanzantseva 2006, p. 48). $\qquad \square$

Lemma 2.4 (Alber 1996) *Let* X *be a reflexive striclty convex and smooth Banach space with* X^* *as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \le V(x, x^* + y^*) \tag{2.10}$$

for all $x \in X$ *and* $x^*, y^* \in X^*$.

Definition 2.5 An operator $T : X \to X^*$ is called *ψ-strongly monotone* if there exists a continuous, strictly increasing function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$ such that

$$\langle Tx - Ty, x - y \rangle \ge \psi(\|x - y\|)\|x - y\| \quad \forall x, y \in D(T). \tag{2.11}$$

Let X and Y be Banach spaces with X^* and Y^* as their respective duals.

Definition 2.6 An operator $A : D(A) \subset X \to Y^*$ is called *hemicontinuous* at $x_0 \in D(A)$ if $x_0 + t_n y \in D(A)$,

$$\text{for } y \in X \text{ and } t_n \to 0_+ \implies A(x_0 + t_n y) \overset{w^*}{\to} Ax_0.$$

Clearly, every continuous map is hemicontinuous.

Lemma 2.7 *Let* $T : X \to X^*$ *be a hemicontinuous ψ-strongly monotone operator with* $D(T) = X$. *Then,* $R(T) = X^*$.

Proof See chapter III, page 48 of Pascali and Sburian (1978). $\qquad \square$

Main results

Convergence in L_p spaces, $1 < p \le 2$.

In the sequel, k is the strong monotonicity constant of A and $L > 0$ is its Lipschitz constant, and $\delta := \frac{k}{2(L_1+1)(L+1)^2}$.

Theorem 4.1 *Let $E = L_p$, $1 < p \le 2$. Let $A : E \to E^*$ be a strongly monotone and Lipschitz map. For $x_0 \in E$ arbitrary, let the sequence $\{x_n\}$ be defined by:*

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad n \ge 0, \tag{4.1}$$

where $\lambda \in \left(0, \delta\right)$. Then, the sequence $\{x_n\}$ converges strongly to $x^* \in A^{-1}(0)$ and x^* is unique.

Proof Let $\psi(t) = kt$ in inequality (2.11). By Lemma 2.7, $A^{-1}(0) \ne \emptyset$. Let $x^* \in A^{-1}(0)$. Using the definition of x_{n+1} we compute as follows:

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(Jx_n - \lambda Ax_n)) \\
&= V(x^*, Jx_n - \lambda Ax_n)
\end{aligned}$$

Applying Lemma 2.4, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= V(x^*, Jx_n - \lambda Ax_n) \\
&\le V(x^*, Jx_n) - 2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - x^*, Ax_n - Ax^*\rangle \\
&= \phi(x^*, x_n) - 2\lambda\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&\quad + 2\lambda\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&\quad - 2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - x^*, Ax_n - Ax^*\rangle \\
&= \phi(x^*, x_n) - 2\lambda\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&\quad - 2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n), Ax_n - Ax^*\rangle.
\end{aligned}$$

Using the strong monotonocity of A, Lipschitz property of j^{-1} and the Lipschitz property of A, we have that:

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\le \phi(x^*, x_n) - 2\lambda k\|x_n - x^*\|^2 \\
&\quad + 2\lambda\|J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n)\|\|Ax_n - Ax^*\| \\
&\le \phi(x^*, x_n) - 2\lambda k\|x_n - x^*\|^2 + 2\lambda^2 L_1 L^2\|x_n - x^*\|^2 \\
&\le \phi(x^*, x_n) - \lambda k\|x_n - x^*\|^2.
\end{aligned}$$

Thus, $\phi(x^*, x_n)$ converges, since it is monotone decreasing and bounded below by zero. Consequently,

$$\lambda k\|x_n - x^*\|^2 \le \phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0, \quad as\ n \to \infty.$$

This yields $x_n \to x^*$ as $n \to \infty$. Suppose there exists $y^* \in A^{-1}(0)$, $y^* \ne x^*$. Then, substituting x^* by y^* in the above argument, we obtain that $x_n \to y^*$ as $n \to \infty$. By uniqueness of limit $x^* = y^*$. So, x^* is unique. completing the proof. $\square$

Convergence in L_p spaces, $2 \le p < \infty$.

Remark 1 We remark that for $E = L_p$, $2 \le p < \infty$, if $A : E \to E^*$ satisfies the following conditions: there exists $k \in (0, 1)$ such that

$$\langle Ax - Ay, x - y\rangle \ge k\|x - y\|^{\frac{p}{p-1}} \quad \forall\, x, y \in E, \tag{5.1}$$

and $A^{-1}(0) \neq \emptyset$, then the *Krasnoselskii-type sequence* (4.1) converges strongly to the unique solution of $Au = 0$. In fact, we prove the following theorem.

In the following theorem, $\delta_p := \left(\frac{k}{2m_p L^{\frac{p}{p-1}}} \right)^{p-1}$.

Theorem 5.1 *Let $X = L_p$, $2 \leq p < \infty$. Let $A : X \to X^*$ be a Lipschitz map. Assume that there exists a constant $k \in (0, 1)$ such that A satisfies the following condition:*

$$\langle Ax - Ay, x - y \rangle \geq k \|x - y\|^{\frac{p}{p-1}}, \tag{5.2}$$

and that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_0 \in X$, define the sequence $\{x_n\}$ iteratively by:

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad n \geq 0, \tag{5.3}$$

where $\lambda \in (0, \delta_p)$. Then, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

Proof We first prove that $\{x_n\}$ is bounded. This proof is by induction.

Let $x^* \in A^{-1}(0)$. Then, there exists $r > 0$ such that $\phi(x^*, x_1) \leq r$. By construction, $\phi(x^*, x_1) \leq r$. Suppose that $\phi(x^*, x_n) \leq r$, for some $n \geq 1$. We prove that $\phi(x^*, x_{n+1}) \leq r$. Using Eq (2.6) and inequality (2.10), we have:

$$\begin{aligned}
\phi(x^*, x_{n+1}) = \phi(x^*, J^{-1}(Jx_n - \lambda Ax_n)) &= V(x^*, Jx_n - \lambda Ax_n) \\
&\leq V(x^*, Jx_n) - 2\langle J^{-1}(Jx_n - \lambda Ax_n) - x^*, \lambda Ax_n \rangle \\
&= V(x^*, Jx_n) - 2\lambda \langle x_n - x^*, Ax_n - Ax^* \rangle \\
&\quad + 2\lambda \langle J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n), Ax_n - Ax^* \rangle. \\
&\leq \phi(x^*, x_n) - 2\lambda \langle x_n - x^*, Ax_n - Ax^* \rangle \\
&\quad + 2\lambda \|J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n)\| \|Ax_n - Ax^*\|.
\end{aligned}$$

Using condition (5.2) on A and inequality (2.8), we obtain:

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \phi(x^*, x_n) - 2k\lambda \|x_n - x^*\|^{\frac{p}{p-1}} + 2\lambda \lambda^{\frac{1}{p-1}} m_p \|Ax_n\|^{\frac{1}{p-1}} \|Ax_n - Ax^*\| \\
&\leq \phi(x^*, x_n) - 2k\lambda \|x_n - x^*\|^{\frac{p}{p-1}} + 2\lambda \lambda^{\frac{1}{p-1}} m_p \|Ax_n - Ax^*\|^{\frac{p}{p-1}}. \\
&\leq \phi(x^*, x_n) - 2k\lambda \|x_n - x^*\|^{\frac{p}{p-1}} + 2\lambda \lambda^{\frac{1}{p-1}} m_p L^{\frac{p}{p-1}} \|x_n - x^*\|^{\frac{p}{p-1}} \\
&\leq \phi(x^*, x_n) - k\lambda \|x_n - x^*\|^{\frac{p}{p-1}} \\
&\leq r.
\end{aligned}$$

Hence, by induction, $\{x_n\}$ is bounded. We now prove that $\{x_n\}$ converges strongly to $x^* \in A^{-1}(0)$. Let $x^* \in A^{-1}(0)$. From the same computation as above, we have that:

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - \lambda k \|x_n - x^*\|^{\frac{p}{p-1}},$$

which implies $\phi(x^*, x_n)$ is decreasing and bounded below by zero, so the limit of $\phi(x^*, x_n)$ exists. Therefore,

$$0 \leq \lim \left(\lambda k \|x_n - x^*\|^{\frac{p}{p-1}} \right) \leq \lim \left(\phi(x^*, x_n) - \phi(x^*, x_{n+1}) \right) = 0.$$

Hence, $x_n \to x^*$ as $n \to \infty$. Uniqueness follows as in the proof of Theorem 4.1. □

Open Question

If $E = L_p$, $2 \leq p < \infty$, attempts to obtain strong convergence of the *Krasnoselskii-type sequence* defined for $x_0 \in E$, by:

$$x_{n+1} = J^{-1}(J(x_n) - \lambda A x_n), \quad n \geq 0, \lambda \in (0, 1) \tag{5.4}$$

to a solution of the equation $Au = 0$, where A is strongly monotone and Lipschitz, have not yielded any positive result. It is, therefore, of interest to find out if a *Krasnoselskii-type sequence* will converge strongly to a solution of $Au = 0$ *in this space*.

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Compliance with ethical guidelines

References

Alber YI (1996) Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos AG (ed) Theory and applications of nonlinear operators of accretive and monotone type. Marcel Dekker, New York, pp. 15–50

Alber YI, Guerre-Delabriere S (2001) On the projection methods for fixed point problems. Analysis (Munich). 21(1):17–39

Alber YI, Ryazantseva I (2006) Nonlinear ill posed problems of monotone type, Springer, London

Berinde V (2007) Iterative approximation of fixed points, lecture notes in mathematics, Springer, London

Browder FE (1967) Nonlinear mappings of nonexpansive and accretive type in Banach spaces. Bull Am Math Soc 73:875–882

Bynum WL (1976) Weak parallelogram laws for Banach spaces. Can Math Bull 19(3):269–275

Chidume CE (1986) An approximation method for monotone Lipschitzian operators in Hilbert-spaces. J Aust Math Soc Ser Pure Math Stat 41:59–63

Chidume CE (1987) Iterative approximation of fixed points of Lipschitzian strictly pseudocontractive mappings. Proc Am Math Soc 99(2):283–288

Chidume CE (1990) Iterative solution of nonlinear equations of the monotone type in Banach spaces. Bull Aust Math Soc 42:21–31

Chidume CE, Osilike MO (1999) Iterative solutions of nonlinear accretive operator equations in arbitrary Banach spaces. Nonlinear Anal Theo Methods Appl 36:863–872

Chidume CE (2002) Convergence theorems for asymptotically pseudocontractive mappings. Nonlinear Anal Theo Methods Appl 49:1–11

Chidume CE, Chidume CO (2005) Convergence theorems for fixed points of uniformly continuous generalized Phi-hemi-contractive mappings. J Math Anal Appl 303:545–554

Chidume CE, Chidume CO (2006) Convergence theorem for zeros of generalized Phi-quasi-accretive operators. Proc Am Math Soc 134:243–251

Chidume CE, Ali B (2007) Approximation of common fixed points for finite families of nonself asymptotically nonexpansive mappings in Banach spaces. J Math Anal Appl 326:960–973

Chidume CE (2009) Geometric properties of Banach spaces and nonlinear iterations, lectures notes in mathematics. vol 1965, Springer, London

Cioranescu I (1990) Geometry of Banach spaces, duality mappings and nonlinear problems. vol 62, Kluwer, Dordrecht

Deng L (1993) On Chidume's open question. J Math Appl 174(2):441–449

Deng L (1993) An iterative process for nonlinear Lipschitz strongly accretive mappings in uniformly convex and uniformly smooth Banach spaces. Acta Appl Math 32(2):183–196

Diemling K (1985) Nonlinear functional analysis, Springer, New York

Goebel K, Reich S (1984) Uniform convexity, hyperbolic geometry, and nonexpansive mappings. Monographs and text-books in pure and applied mathematics. vol. 83, Marcel Dekker inc., New York

Iiduka H, Takahashi W, Toyoda M (2004) Approximation of solutions of variational inequalities for monotone mappings. Panamer Math J 14:49–61

Kamimura S, Takahashi W (2002) Strong convergence of a proximal-type algorithm in a Banach space. SIAMJ Optim 13(3):938–945

Liu L (1997) Approximation of fixed points of a strictly pseudocontractive mapping. Proc Am Math Soc 125(5):1363–1366

Liu L (1995) Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J Math Anal Appl 194(1):114–125

Pascali D, Sburian S (1978) Nonlinear mappings of monotone type. Editura Academia Bucaresti, Romania

Qihou L (2002) Iterative sequences for asymptotically quasi-nonexpansive mapping with an error member of uniformly convex Banach space. J Math Anal Appl 266:468–471

Qihou L (1990) The convergence theorems of the sequence of Ishikawa iterates for hemi-contractive mapping. J Math Anal Appl 148:55–62

Reich S (1996) A weak convergence theorem for the alternating methods with Bergman distance. In: Kartsatos AG (ed) Theory and applications of nonlinear operators of accretive and monotone type, in lecture notes in pure and Appl Math, vol 178 Dekker, New York, pp 313–318

Weng XL (1991) Fixed point iteration for local striclty pseudocontractive mappings. Proc Am Math Soc 113(3):727–731

Weng XL (1992) Iterative construction of fixed points of a dissipative type operator. Tamkang J Math 23:205–215

William K, Shahzad N (2015) Fixed point theory in distance spaces, Springer, New York

Xiao R (1998) Chidume's open problems and fixed point theorems. Xichuan Daxue Xuebao 35(4):505–508

Xu ZB (1989) Characteristic inequalities of L_p spaces and their applications. Acta Math Sinica 32(2):209–218

Xu ZB (1992) A note on the Ishikawa iteration schemes. J Math Anal Appl 167:582–587

Xu ZB, Roach GF (1991) Characteristic inequalities for uniformly convex and uniformly smooth Banach space. J Math Anal Appl 157:189–210

Xu ZB, Roach GF (1992) A necessary and sufficient condition for convergence for of a steepest descent approximation to accretive operator equations. J Math Anal Appl 167:340–354

Xu ZB, Jiang YL, Roach GF (1995) A further necessary and sufficient condition for strong convergence of nonlinear contraction semigroups and of iteration methods for accretive operators in Banach spaces. Proc Edinburgh Math Soc 38(2):1–12

Xu HK (1991) Inequalities in Banach spaces with applications. Nonlinear Anal 16(12):1127–1138

Xu Y (1991) Existence and convergence for fixed points of mappings of the asymptotically nonexpansive type. Nonlinear Anal 16:1139–1146

Xu Y (1998) Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations. J Math Anal Appl 224:91–101

Zegeye H, Shahzad N (2009) Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings. Nonlinear Anal 70:2707–2716

Zhou H (1997) Iterative solutions of nonlinear equations involving strongly accretive operators without the Lipschitz assumption. J Math Anal Appl 213(1):296–307

Zhou H, Jia Y (1996) Approximating the zeros of accretive operators by the Ishikawa iteration process. Abstr Appl Anal 1(2):153–167

Zhou H, Jia Y (1997) Approximation of fixed points of strongly pseudocontractive maps without Lipschitz assumption. Proc Am Math Soc 125(6):1705–1709

Zhu L (1994) Iteration solution of nonlinear equations involving m-accretive operators in Banach spaces. J Math Anal Appl 188:410–415