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A characterization of Chover-type law of iterated logarithm

Deli Li[1] and Pingyan Chen[2*]

Abstract

Let $0 < \alpha \leq 2$ and $-\infty < \beta < \infty$. Let $\{X_n; n \geq 1\}$ be a sequence of independent copies of a real-valued random variable X and set $S_n = X_1 + \cdots + X_n$, $n \geq 1$. We say X satisfies the (α, β)-Chover-type law of the iterated logarithm (and write $X \in CTLIL(\alpha, \beta)$) if $\limsup_{n\to\infty} \left|\frac{S_n}{n^{1/\alpha}}\right|^{(\log\log n)^{-1}} = e^{\beta}$ almost surely. This paper is devoted to a characterization of $X \in CTLIL(\alpha, \beta)$. We obtain sets of necessary and sufficient conditions for $X \in CTLIL(\alpha, \beta)$ for the five cases: $\alpha = 2$ and $0 < \beta < \infty$, $\alpha = 2$ and $\beta = 0$, $1 < \alpha < 2$ and $-\infty < \beta < \infty$, $\alpha = 1$ and $-\infty < \beta < \infty$, and $0 < \alpha < 1$ and $-\infty < \beta < \infty$. As for the case where $\alpha = 2$ and $-\infty < \beta < 0$, it is shown that $X \notin CTLIL(2, \beta)$ for any real-valued random variable X. As a special case of our results, a simple and precise characterization of the classical Chover law of the iterated logarithm (i.e., $X \in CTLIL(\alpha, 1/\alpha)$) is given; that is, $X \in CTLIL(\alpha, 1/\alpha)$ if and only if $\inf\left\{b : \mathbb{E}\left(\frac{|X|^{\alpha}}{(\log(e\vee|X|))^{b\alpha}}\right) < \infty\right\} = 1/\alpha$ where $\mathbb{E}X = 0$ whenever $1 < \alpha \leq 2$.

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1 Introduction

Throughout, $\{X_n;\ n \geq 1\}$ is a sequence of independent copies of a real-valued random variable X. As usual, the partial sums of independent identically distributed (i.i.d.) random variables X_n, $n \geq 1$ will be denoted by $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. Write $Lx = \log(e \vee x)$, $x \geq 0$.

When X has a symmetric stable distribution with exponent $\alpha \in (0, 2)$, i.e., $\mathbb{E}\left(e^{itX}\right) = e^{-|t|^{\alpha}}$ for $t \in (-\infty, \infty)$, Chover 1966 proved that

$$\limsup_{n\to\infty} \left|\frac{S_n}{n^{1/\alpha}}\right|^{(\log\log n)^{-1}} = e^{1/\alpha} \text{ almost surely (a.s.).} \tag{1.1}$$

This is what we call the classical Chover law of iterated logarithm (LIL). Since then, several papers have been devoted to develop the classical Chover LIL. See, for example, Hedye 1969 showed that (1.1) holds when X is in the domain of normal attraction of a nonnormal stable law, Pakshirajan and Vasudeva 1977 discussed the limit points of the sequence $\{|S_n/n^{1/\alpha}|^{(\log\log n)^{-1}}; n \geq 2\}$, Kuelbs and Kurtz 1974 obtained the classical Chover LIL in a Hilbert space setting, Chen 2002 obtained the classical Chover LIL for the weighed sums, Vasudeva 1984, Qi and Cheng 1996, Peng and Qi 2003 established the Chover LIL when X is in the domain of attraction of a nonnormal stable law, Scheffler 2000 studied the classical Chover LIL when X is in the generalized domain of operator semistable attraction of some nonnormal law, Chen and Hu 2012 extended the results of Kuelbs and Kurtz 1974 to an arbitrary real separable Banach space, and so on. It should be pointed out that the previous papers only gave sufficient conditions for the classical Chover LIL.

Motivated by the previous study of the classical Chover LIL, we introduce a general Chover-type LIL as follows.

Definition 1.1. *Let $0 < \alpha \leq 2$ and $-\infty < \beta < \infty$. Let $\{X, X_n;\ n \geq 1\}$ be a sequence of real-valued i.i.d. random*

*Correspondence: tchenpy@jnu.edu.cn
[2]Department of Mathematics, Jinan University, Guangzhou 510630, China
Full list of author information is available at the end of the article

variables. We say X satisfies the (α, β)-Chover-type law of the iterated logarithm (and write $X \in CTLIL(\alpha, \beta)$) if

$$\limsup_{n\to\infty} \left| \frac{S_n}{n^{1/\alpha}} \right|^{(\log\log n)^{-1}} = e^{\beta} \ \ a.s. \tag{1.2}$$

From the classical Chover LIL and Definition 1.1, we see that $X \in CTLIL(\alpha, 1/\alpha)$ (i.e., (1.2) holds with $\beta = 1/\alpha$) when X has a symmetric stable distribution with exponent $\alpha \in (0, 2)$.

This paper is devoted to a characterization of $X \in CTLIL(\alpha, \beta)$. The main results are stated in Section 2. We obtain sets of necessary and sufficient conditions for $X \in CTLIL(\alpha, \beta)$ for the five cases: $\alpha = 2$ and $0 < \beta < \infty$ (see Theorem 2.1), $\alpha = 2$ and $\beta = 0$ (see Theorem 2.2), $1 < \alpha < 2$ and $-\infty < \beta < \infty$ (see Theorem 2.3), $\alpha = 1$ and $-\infty < \beta < \infty$ (see Theorem 2.4), and $0 < \alpha < 1$ and $-\infty < \beta < \infty$ (see Theorem 2.5). The proofs of Theorems 2.1-2.5 are given in Section 4. For proving Theorems 2.1-2.5, three preliminary lemmas are stated in Section 3. Some llustrative examples are provided in Section 5.

2 Statement of the main results

The main results of this paper are the following five theorems. We begin with the case where $\alpha = 2$ and $0 < \beta < \infty$.

Theorem 2.1. *Let $0 < \beta < \infty$. Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. real-valued random variables. Then*

$$X \in CTLIL(2, \beta), \ i.e., \ \limsup_{n\to\infty} \left| \frac{S_n}{\sqrt{n}} \right|^{(\log\log n)^{-1}} = e^{\beta} \ \ a.s. \tag{2.1}$$

if and only if

$$\mathbb{E}X = 0 \ \ and \ \ \inf\left\{ b > 0 : \ \mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}} \right) < \infty \right\} = \beta. \tag{2.2}$$

For the case where $\alpha = 2$ and $\beta = 0$, we have the following result.

Theorem 2.2. *Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. non-degenerate real-valued random variables. Then*

$$\limsup_{n\to\infty} \left| \frac{S_n}{\sqrt{n}} \right|^{(\log\log n)^{-1}} \leq 1 \ \ a.s. \tag{2.3}$$

if and only if

$$\mathbb{E}X = 0 \ \ and \ \ \inf\left\{ b > 0 : \ \mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}} \right) < \infty \right\} = 0. \tag{2.4}$$

In either case, we have

$$X \in CTLIL(2, 0), \ i.e., \ \limsup_{n\to\infty} \left| \frac{S_n}{\sqrt{n}} \right|^{(\log\log n)^{-1}} = 1 \ \ a.s. \tag{2.5}$$

Remark 2.1. *Let c be a constant. Note that*

$$\limsup_{n\to\infty} \left| \frac{nc}{\sqrt{n}} \right|^{(\log\log n)^{-1}} = \begin{cases} 0 & if \ c = 0, \\ \infty & if \ c \neq 0. \end{cases}$$

Thus, from Theorem 2.2, we conclude that, for any $-\infty < \beta < 0$, $X \notin CTLIL(2, \beta)$ for any real-valued random variable X.

In the next three theorems, we provide necessary and sufficient conditions for $X \in CTLIL(\alpha, \beta)$ for the three cases where $1 < \alpha < 2$ and $-\infty < \beta < \infty$, $\alpha = 1$ and $-\infty < \beta < \infty$, and $0 < \alpha < 1$ and $-\infty < \beta < \infty$ respectively.

Theorem 2.3. *Let $1 < \alpha < 2$ and $-\infty < \beta < \infty$. Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. real-valued random variables. Then*

$$X \in CTLIL(\alpha, \beta), \ i.e., \ \limsup_{n\to\infty} \left| \frac{S_n}{n^{1/\alpha}} \right|^{(\log\log n)^{-1}} = e^{\beta} \ \ a.s.$$

if and only if

$$\mathbb{E}X = 0 \ \ and \ \ \inf\left\{ b : \ \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}} \right) < \infty \right\} = \beta.$$

Theorem 2.4. *Let $-\infty < \beta < \infty$. Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. real-valued random variables. Then*

$$X \in CTLIL(1, \beta), \ i.e., \ \limsup_{n\to\infty} \left| \frac{S_n}{n} \right|^{(\log\log n)^{-1}} = e^{\beta} \ \ a.s.$$

if and only if

$$\begin{cases} \inf\left\{ b : \mathbb{E}\left(\frac{|X|}{(L|X|)^{b}} \right) < \infty \right\} = \beta & if \ \beta > 0, \\ either \ \ \mathbb{E}|X| < \infty \ and \ \mathbb{E}X \neq 0 & \\ or \ \ \inf\left\{ b : \mathbb{E}\left(\frac{|X|}{(L|X|)^{b}} \right) < \infty \right\} = 0 & if \ \beta = 0, \\ \mathbb{E}X = 0 \ and \ \inf\left\{ b : \mathbb{E}\left(\frac{|X|}{(L|X|)^{b}} \right) < \infty \right\} = \beta & if \ \beta < 0. \end{cases}$$

In particular, $\mathbb{E}|X| < \infty$ *and* $\mathbb{E}X \neq 0$ *imply that*

$$\lim_{n\to\infty}\left|\frac{S_n}{n}\right|^{(\log\log n)^{-1}} = 1 \ \ a.s.$$

Theorem 2.5. *Let* $0 < \alpha < 1$ *and* $-\infty < \beta < \infty$*. Let* $\{X, X_n; n \geq 1\}$ *be a sequence of i.i.d. real-valued random variables. Then*

$$X \in CTLIL(\alpha,\beta), \ i.e., \ \limsup_{n\to\infty}\left|\frac{S_n}{n^{1/\alpha}}\right|^{(\log\log n)^{-1}} = e^{\beta} \ \ a.s.$$

if and only if

$$\inf\left\{b : \ \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) < \infty\right\} = \beta.$$

Remark 2.2. *From our Theorems 2.1, 2.3, 2.4, and 2.5, a simple and precise characterization of the classical Chover LIL (i.e.,* $X \in CTLIL(\alpha, 1/\alpha)$*) is obtained as follows. For* $0 < \alpha \leq 2$*, we have*

$$X \in CTLIL(\alpha,1/\alpha), \ i.e., \ \limsup_{n\to\infty}\left|\frac{S_n}{n^{1/\alpha}}\right|^{(\log\log n)^{-1}} = e^{1/\alpha} \ \ a.s.$$

if and only if

$$\inf\left\{b : \ \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) < \infty\right\} = 1/\alpha \ \ where \ \ \mathbb{E}X = 0 \ \ whenever \ \ 1 < \alpha \leq 2.$$

Our Theorems 2.1-2.5 also imply the following two interesting results.

Corollary 2.1. *Let* $0 < \alpha \leq 2$*. Let* $\{X, X_n; n \geq 1\}$ *be a sequence of i.i.d. real-valued random variables. Then*

$$\lim_{n\to\infty}\left|\frac{S_n}{n^{1/\alpha}}\right|^{(\log\log n)^{-1}} = 0 \ \ a.s.$$

if and only if

$$\begin{cases} \inf\left\{b : \ \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) < \infty\right\} = -\infty & if\, 0 < \alpha < 1, \\ \mathbb{E}X = 0 \ and \ \inf\left\{b : \ \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) < \infty\right\} = -\infty & if\, 1 \leq \alpha < 2, \\ X = 0 \ \ a.s. & if\, \alpha = 2. \end{cases}$$

Corollary 2.2. *Let* $0 < \alpha \leq 2$*. Let* $\{X, X_n; n \geq 1\}$ *be a sequence of i.i.d. real-valued random variables. Then*

$$\limsup_{n\to\infty}\left|\frac{S_n}{n^{1/\alpha}}\right|^{(\log\log n)^{-1}} = \infty \ \ a.s.$$

if and only if

$$\begin{cases} \inf\left\{b : \ \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) < \infty\right\} = \infty & if\, 0 < \alpha \leq 1, \\ eithor\ \mathbb{E}X \neq 0 \ or \ \inf\left\{b : \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) < \infty\right\} = \infty & if\, 1 < \alpha \leq 2. \end{cases}$$

From our main results Theorems 2.1-2.5 and Corollaries 2.1-2.2 above, an almost unified characterization for $0 < \alpha \leq 2$ stated as the following result was so kindly presented to us by a referee.

Theorem 2.6. *Let* $0 < \alpha \leq 2$*. Let* $\{X, X_n; n \geq 1\}$ *be a sequence of i.i.d. real-valued random variables. Assume that* $\mathbb{E}(X^2) = \infty$ *and* $\mathbb{E}X = 0$ *whenever* $\mathbb{E}|X| < \infty$*. Then*

$$\limsup_{n\to\infty}\left|\frac{S_n}{n^{1/\alpha}}\right|^{(\log\log n)^{-1}} = e^{\beta} \ \ a.s.$$

where

$$\beta = \inf\left\{b : \ \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) < \infty\right\} \in [-\infty, \infty].$$

3 Preliminary lemmas

To prove the main results, we use the following three preliminary lemmas. The first lemma is new and may be of independent interest.

Lemma 3.1. *Let* $\{a_n; \ n \geq 1\}$ *be a sequence of real numbers. Let* $\{c_n; \ n \geq 1\}$ *be a sequence of positive real numbers such that*

$$\lim_{n\to\infty} c_n = \infty. \tag{3.1}$$

Then we have

(i) There exists a constant $-\infty < \beta < \infty$ *such that*

$$\limsup_{n\to\infty} |a_n|^{1/c_n} = e^{\beta} \tag{3.2}$$

if and only if

$$\limsup_{n\to\infty}\frac{|a_n|}{e^{bc_n}} = \begin{cases} 0 & for\ all\ b > \beta, \\ \infty & for\ all\ b < \beta; \end{cases} \tag{3.3}$$

(ii) There exists a constant $-\infty < \beta < \infty$ *such that*

$$\liminf_{n\to\infty} |a_n|^{1/c_n} = e^{\beta}$$

if and only if

$$\liminf_{n\to\infty}\frac{|a_n|}{e^{bc_n}} = \begin{cases} 0 & for\ all\ b > \beta, \\ \infty & for\ all\ b < \beta; \end{cases}$$

(iii) There exists a constant $-\infty < \beta < \infty$ *such that*

$$\lim_{n\to\infty} |a_n|^{1/c_n} = e^{\beta}$$

if and only if

$$\lim_{n\to\infty}\frac{|a_n|}{e^{bc_n}}=\begin{cases}0 & \text{for all } b>\beta,\\ \infty & \text{for all } b<\beta.\end{cases}$$

Proof We prove the sufficiency part of Part (i) first. It follows from (3.3) that the set

$$\left\{n\ge 1;\ \frac{|a_n|}{e^{bc_n}}>1\right\}\begin{cases}\text{has finitely many elements} & \text{if } b>\beta,\\ \text{has infinitely many elements} & \text{if } b<\beta.\end{cases}$$

Note that

$$\left\{n\ge 1;\ |a_n|^{1/c_n}>e^b\right\}=\left\{n\ge 1;\ \frac{|a_n|}{e^{bc_n}}>1\right\}.$$

We thus conclude that

$$\limsup_{n\to\infty}|a_n|^{1/c_n}\begin{cases}\le e^b & \text{for all } b>\beta,\\ \ge e^b & \text{for all } b<\beta\end{cases}$$

which ensures (3.2).

We now prove the necessity part of Part (i). For all $b\neq\beta$, let $h=(b+\beta)/2$. Then

$$\begin{cases}\beta<h<b & \text{if } b>\beta,\\ b<h<\beta & \text{if } b<\beta.\end{cases}$$

It follows from (3.2) that the set

$$\left\{n\ge 1;\ |a_n|^{1/c_n}>e^h\right\}\begin{cases}\text{has finitely many elements} & \text{if } b>\beta,\\ \text{has infinitely many elements} & \text{if } b<\beta.\end{cases}$$

Note that

$$\left\{n\ge 1;\ \frac{|a_n|}{e^{hc_n}}>1\right\}=\left\{n\ge 1;\ |a_n|^{1/c_n}>e^h\right\}.$$

We thus have that

$$\limsup_{n\to\infty}\frac{|a_n|}{e^{hc_n}}\begin{cases}\le 1 & \text{if } b>\beta,\\ \ge 1 & \text{if } b<\beta.\end{cases}\tag{3.4}$$

Note that (3.1) implies that

$$\lim_{n\to\infty}\frac{e^{hc_n}}{e^{bc_n}}=\lim_{n\to\infty}e^{(h-b)c_n}=\begin{cases}0 & \text{if } b>\beta,\\ \infty & \text{if } b<\beta.\end{cases}$$

Thus it follows from (3.4) that

$$\limsup_{n\to\infty}\frac{|a_n|}{e^{bc_n}}=\left(\lim_{n\to\infty}\frac{e^{hc_n}}{e^{bc_n}}\right)\left(\limsup_{n\to\infty}\frac{|a_n|}{e^{hc_n}}\right)=\begin{cases}0 & \text{if } b>\beta,\\ \infty & \text{if } b<\beta,\end{cases}$$

i.e., (3.3) holds.

We leave the proofs of Parts (ii) and (iii) to the reader since they are similar to the proof of Part (i). This completes the proof of Lemma 3.1. □

The following result is a special case of Corollary 2 of Einmahl and Li 2005.

Lemma 3.2. *Let $b>0$. Let $\{X,X_n;n\ge 1\}$ be a sequence of i.i.d. random variables. Then*

$$\lim_{n\to\infty}\frac{S_n}{\sqrt{n}(\log n)^b}=0\ \ a.s.$$

if and only if

$$\mathbb{E}X=0,\ \mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}}\right)<\infty,\ \text{and}\ \lim_{x\to\infty}\frac{LLx}{(Lx)^{2b}}H(x)=0,$$

where $H(x)=\mathbb{E}\left(X^2I\{|X|\le x\}\right)$, $x\ge 0$.

The following result is a generalization of Kolmogorov-Marcinkiewicz-Zygmund strong law of large numbers and follows easily from Theorems 1 and 2 of Feller 1946.

Lemma 3.3. *Let $0<\alpha<2$ and $-\infty<b<\infty$. Let $\{X,X_n;n\ge 1\}$ be a sequence of i.i.d. random variables. Then*

$$\lim_{n\to\infty}\frac{S_n}{n^{1/\alpha}(\log n)^b}=0\ \ a.s.$$

if and only if

$$\mathbb{E}\left(\frac{|X|^\alpha}{(L|X|)^{b\alpha}}\right)<\infty$$

where $\mathbb{E}X=0$ whenever either $1<\alpha<2$ or $\alpha=1$ and $-\infty<b\le 0$.

4 Proofs of the main results

In this section, we only give the proofs of Theorems 2.1-2.2. By applying Lemmas 3.1 and 3.3, the proofs of Theorems 2.3-2.5 involves only minor modifications of the proof of Theorem 2.1 and will be omitted.

Proof of Theorem 2.1 We prove the sufficiency part first. Note that the second part of (2.2) implies that

$$\mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}}\right)<\infty\ \text{ for all } b>\beta.\tag{4.1}$$

We thus see that for all $b>\beta$,

$$\begin{aligned}H(x)&=\mathbb{E}\left(X^2I\{|X|\le x\}\right)\\ &\le\mathbb{E}\left(\frac{X^2}{(L|X|)^{2h}}(Lx)^{2h}I\{|X|\le x\}\right)\\ &\le (Lx)^{2h}\mathbb{E}\left(\frac{X^2}{(L|X|)^{2h}}\right),\end{aligned}$$

where $h=(b+\beta)/2$. Since $\beta<h<b$, it follows from (4.1) that

$$\frac{LLx}{(Lx)^{2b}}H(x)\le\frac{LLx}{(Lx)^{2(b-h)}}\mathbb{E}\left(\frac{X^2}{(L|X|)^{2h}}\right)\to 0\ \text{ as } x\to\infty.$$

We thus conclude from (2.2) that

$$\mathbb{E}X = 0, \ \mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}}\right) < \infty, \ \text{and}$$
$$\lim_{x\to\infty} \frac{LLx}{(Lx)^{2b}} H(x) = 0 \ \text{ for all } b > \beta > 0$$

which, by applying Lemma 3.2, ensures that

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}(\log n)^b} = 0 \ \text{ a.s. } \ \text{ for all } b > \beta. \tag{4.2}$$

Since $\beta > 0$, the second part of (2.2) implies that

$$\mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}}\right) = \infty \ \text{ for all } b < \beta$$

which ensures that

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{n}(\log n)^b} = \infty \ \text{ a.s. } \ \text{ for all } b < \beta. \tag{4.3}$$

Let $A_n = S_n/\sqrt{n}$ and $c_n = LLn$, $n \geq 1$. It then follows from (4.2) and (4.3) that

$$\limsup_{n\to\infty} \frac{|A_n|}{e^{bc_n}} = \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{n}(\log n)^b} = \begin{cases} 0 \text{ a.s. for all } b > \beta, \\ \infty \text{ a.s. for all } b < \beta. \end{cases} \tag{4.4}$$

By Lemma 3.1, we see that (4.4) is equivalent to

$$\limsup_{n\to\infty} |A_n|^{1/c_n} = e^{\beta} \ \text{ a.s.,}$$

i.e., (2.1) holds.

We now prove the necessity part. By Lemma 3.1, (2.1) is equivalent to (4.4) which ensures that (4.2) holds. By Lemma 3.2, we conclude from (4.2) that

$$\mathbb{E}X = 0 \ \text{ and } \ \mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}}\right) < \infty \ \text{ for all } b > \beta. \tag{4.5}$$

Since $0 < \beta < \infty$, it follows from (4.5) that

$$\beta_1 \triangleq \inf\left\{b > 0 : \ \mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}}\right) < \infty\right\} \leq \beta.$$

If $\beta_1 < \beta$ then, using the argument in the proof of the sufficiency part, we have that

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}(\log n)^b} = 0 \ \text{ a.s. } \ \text{ for all } b > \beta_1. \tag{4.6}$$

Hence, by Lemma 3.1, (4.6) implies that

$$\limsup_{n\to\infty} \left|\frac{S_n}{\sqrt{n}}\right|^{(\log\log n)^{-1}} \leq e^{\beta_1} < e^{\beta} \ \text{ a.s.}$$

which is in contradiction to (2.1). Thus (2.2) holds. The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2 Using the same argument used in the proof of the sufficiency part of Theorem 2.1, we have from (2.4) that

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}(\log n)^b} = 0 \ \text{ a.s. } \ \text{ for all } b > 0. \tag{4.7}$$

Since X is a non-degenerate random variable, by the classical Hartman-Wintner-Strassen LIL, we have that

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n\log\log n}} > 0 \ \text{ a.s.}$$

which implies that

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{n}(\log n)^b} = \infty \ \text{ a.s. } \ \text{ for all } b < 0. \tag{4.8}$$

Let $A_n = S_n/\sqrt{n}$ and $c_n = LLn$, $n \geq 1$. It then follows from (4.7) and (4.8) that

$$\limsup_{n\to\infty} \frac{|A_n|}{e^{bc_n}} = \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{n}(\log n)^b} = \begin{cases} 0 \ \text{ a.s. for all } b > 0, \\ \infty \ \text{ a.s. for all } b < 0. \end{cases} \tag{4.9}$$

By Lemma 3.1, we see that (4.9) is equivalent to

$$\limsup_{n\to\infty} |A_n|^{1/c_n} = e^0 = 1 \ \text{ a.s.,}$$

i.e., (2.5) holds, so does (2.3).

Using the same argument used in the proof of the necessity part of Theorem 2.1, we conclude from (2.3) that

$$\mathbb{E}X = 0 \ \text{ and } \ \inf\left\{b > 0 : \ \mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}}\right) < \infty\right\} \leq 0.$$

Clearly

$$\inf\left\{b > 0 : \ \mathbb{E}\left(\frac{X^2}{(L|X|)^{2b}}\right) < \infty\right\} \geq 0.$$

Thus (2.4) holds. The proof of Theorem 2.2 is therefore complete. □

5 Examples

In this section, we provide the following examples to illustrate our main results. By applying Theorems 2.3-2.5, we rededuce the classical Chover LIL in the first example.

Example 5.1. *Let $0 < \alpha \leq 2$. Let X be a symmetric real-valued stable random variable with exponent α. Clearly, $\mathbb{E}X = 0$ whenever $1 < \alpha \leq 2$.*

For $0 < \alpha < 2$, we have

$$\mathbb{P}(|X| > x) \sim \left(\frac{\sin(\pi\alpha/2)\,\Gamma(\alpha)}{\pi}\right)|x|^{-\alpha} \ \text{ as } \ x \to \infty,$$

it follows that

$$\mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) \begin{cases} < \infty \ \text{ if } b > 1/\alpha, \\ = \infty \ \text{ if } b \leq 1/\alpha \end{cases}$$

and hence that

$$\inf\left\{b : \ \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) < \infty\right\} = 1/\alpha.$$

Thus, by Theorems 2.3-2.5, $X \in CTLIL(\alpha, 1/\alpha)$ (i.e., the classical Chover LIL follows).

However, for $\alpha = 2$, we have $\mathbb{E}X^2 = 1$. Hence, by Theorems 2.1 and 2.2, we see that $X \notin CTLIL(2, 1/2)$ but $X \in CTLIL(2, 0)$.

From our second example, we will see that $X \in CTLIL(\alpha, \beta)$ for some certain α and β even if the distribution of X is not in the domain of attraction of the stable distribution with exponent α.

Example 5.2. *Let $0 < \alpha \le 2$. Let $d_n = \exp(2^n)$, $n \ge 1$. Given $-\infty < \lambda < \infty$. Let X be a symmetric i.i.d. real-valued random variable such that*

$$\mathbb{P}(X = -d_n) = \mathbb{P}(X = d_n) = \left(\frac{c}{2}\right)\frac{\log^{\lambda} d_n}{d_n^{\alpha}} = \left(\frac{c}{2}\right)\frac{2^{n\lambda}}{d_n^{\alpha}},\ n \ge 1$$

where

$$c = c(\alpha, \lambda) = \left(\sum_{n=1}^{\infty}\frac{\log^{\lambda} d_n}{d_n^{\alpha}}\right)^{-1} > 0.$$

Then

$$\limsup_{x\to\infty}\frac{x^{\alpha}}{\log^{\lambda} x}\mathbb{P}(|X| \ge x) = c > 0 \ \text{and}$$
$$\liminf_{x\to\infty}\frac{x^{\alpha}}{\log^{\lambda} x}\mathbb{P}(|X| \ge x) = 0.$$

Thus the distribution of X is not in the domain of attraction of the stable distribution with exponent α. Also $\mathbb{E}X = 0$ whenever either $1 < \alpha \le 2$ or $\alpha = 1$ and $\lambda < 0$. It is easy to see that

$$\mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) = \left(\frac{c}{2}\right)\sum_{n=1}^{\infty}\left(2^{\lambda - b\alpha}\right)^n \begin{cases} < \infty & \text{if } b > \lambda/\alpha, \\ = \infty & \text{if } b \le \lambda/\alpha \end{cases}$$

and hence that

$$\inf\left\{b : \mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) < \infty\right\} = \lambda/\alpha.$$

Thus, by Theorems 2.1-2.5, we have
(1) If $\alpha = 2$ and $0 < \lambda < \infty$, then $X \in CTLIL(2, \lambda/2)$.
(2) If $\alpha = 2$ and $-\infty < \lambda \le 0$, then $X \in CTLIL(2, 0)$.
(3) If $0 < \alpha < 2$, then $X \in CTLIL(\alpha, \lambda/\alpha)$.

Our third example shows that X may satisfy the other Chover-type LIL studied by Chen and Hu 2012 when $X \notin CTLIL(\alpha, \beta)$.

Example 5.3. *Define the density function $f(x)$ of X by*

$$f(x) = \begin{cases} 0 & \text{if } 0 \le |x| < e, \\ \frac{c}{|x|^{\alpha+1}}\exp\left(p(\log|x|)^{\gamma}\right) & \text{if } |x| \ge e, \end{cases}$$

where $p \ne 0$, $0 < \gamma < 1$, $c = c(\alpha, p, \gamma)$ is a positive constant such that $\int_{-\infty}^{\infty} f(x)dx = 1$. On simplification one can show that for any $-\infty < b < \infty$

$$\mathbb{E}\left(\frac{|X|^{\alpha}}{(L|X|)^{b\alpha}}\right) = 2c\int_e^{\infty}\frac{\exp\left(p(\log x)^{\gamma}\right)}{x(\log x)^{b\alpha}}dx \begin{cases} < \infty & \text{if } p < 0, \\ = \infty & \text{if } p > 0. \end{cases}$$

From Theorem 2.1-2.5, we have
(1) If $\alpha = 2$ and $p < 0$, then $X \in CTLIL(2, 0)$.
(2) If $\alpha = 2$ and $p > 0$, then $X \notin CTLIL(2, \beta)$ for any $0 \le \beta < \infty$.
(3) If $0 < \alpha < 2$, then $X \notin CTLIL(\alpha, \beta)$ for any $-\infty < \beta < \infty$.
However, for $0 < \alpha < 2$, by Theorem 3.1 in Chen and Hu 2012, we have

$$\limsup_{n\to\infty}\left|\frac{S_n}{B(n)}\right|^{(\log\log n)^{-1}} = e^{1/\alpha}\ \ a.s.,$$

where $B(x)$ is the inverse function of $x^{\alpha}/\exp\left(p(\log x)^{\gamma}\right)$, $x \ge e$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DL and PC contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Author details
[1] Department of Mathematical Sciences, Lakehead University, Thunder Bay, P7B 5E1 Ontario, Canada. [2] Department of Mathematics, Jinan University, Guangzhou 510630, China.

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