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A system of nonlinear set valued variational inclusions

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Abstract
In this paper, we studied the existence theorems and techniques for finding the solutions of a system of nonlinear set valued variational inclusions in Hilbert spaces. To overcome the difficulties, due to the presence of a proper convex lower semicontinuous function ϕ and a mapping g which appeared in the considered problems, we have used the resolvent operator technique to suggest an iterative algorithm to compute approximate solutions of the system of nonlinear set valued variational inclusions. The convergence of the iterative sequences generated by algorithm is also proved.

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Introduction

It is well known that variational inequality theory and complementarity problems are very powerful tools of current mathematical technology. In recent years, the classical variational inequality and complementarity problems have been extended and generalized to study a large variety of problems arising in economics, control problems, contact problems, mechanics, transportation, equilibrium problems, optimization theory, nonlinear programming, transportation equilibrium and engineering sciences, see (Aubin 1982; Baiocchi and Capelo 1984; Chang 1984; Giannessi and Maugeri 1995). Hassouni and Moudafi 2001 introduced and studied a class of mixed type variational inequalities with single valued mappings which was called variational inclusions. Since many authors have obtained important extension generalizations of the results in (Hassouni and Moudafi 2001) from various directions, see (Agarwal et al. 2011; Fang et al. 2005; Kassay and Kolumban 2000; Petrot 2010). Verma 1999; 2001a introduced and studied some system of variational inequalities with iterative algorithms to compute approximate solutions in Hilbert spaces.

Inspired and motivated by the research work going on this field, in this works, the methods for finding the common solutions of a system of nonlinear set valued variational inclusions involving different nonlinear operators and fixed point problem are considered and studied, via proximal method in the framework of Hilbert spaces.

Since the problems of a system of a nonlinear set valued variational inequalities and fixed point are both important, the results present in this paper are useful and can be viewed as an improvement and extension of the previously known results appearing in

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literature, which are improves the results of Chang et al. 2007 and also extends the results of Verma 2001b; 2002, Ahmad and Salahuddin 2012, Ding and Luo 2000, Inchan and Petrot 2011, Kim and Kim 2004, Kim and Hu 2008, Nie et al. 2003 and Suantai and Petrot 2011, etc.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively and K be a nonempty closed convex subset of H. Let $CB(H)$ be the family of all nonempty closed convex and bounded sets in H and $\phi : H \to (-\infty, +\infty)$ be a proper convex lower semicontinuous function on H. Let $N_i : H \times H \to H$ be a nonlinear function, $g_i : K \to H$ be a nonlinear operator, $A_i, B_i : K \to CB(H)$ be the nonlinear set valued mappings and let r_i be a fixed positive real number for each $i = 1, 2, 3$. Set $\Xi = \{N_1, N_2, N_3\}, \mathfrak{A} = \{A_1, A_2, A_3\}, \mathfrak{B} = \{B_1, B_2, B_3\}, \wedge = \{g_1, g_2, g_3\}$. *The system of nonlinear set valued variational inclusions involving three different nonlinear operators* is defined as follows:

Find $(x^*, y^*, z^*) \in H \times H \times H,\ u_3^* \in A_3(x^*), v_3^* \in B_3(x^*), u_2^* \in A_2(z^*), v_2^* \in B_2(z^*), u_1^* \in A_1(y^*), v_1^* \in B_1(y^*)$, such that

$$\begin{cases} \left\langle r_1 N_1 \left(u_1^*, v_1^*\right) + g_1(x^*) - g_1(y^*), g_1(x) - g_1(x^*) \right\rangle - r_1\phi(g_1(x^*)) + r_1\phi(g_1(x)) \geq 0,\ \ g_1(x) \in K, \\ \left\langle r_2 N_2 \left(u_2^*, v_2^*\right) + g_2(y^*) - g_2(z^*), g_2(x) - g_2(y^*) \right\rangle - r_2\phi(g_2(y^*)) + r_2\phi(g_2(x)) \geq 0,\ \ g_2(x) \in K, \\ \left\langle r_3 N_3 \left(u_3^*, v_3^*\right) + g_3(z^*) - g_3(x^*), g_3(x) - g_3(z^*) \right\rangle - r_3\phi(g_3(z^*)) + r_3\phi(g_3(x)) \geq 0,\ \ g_3(x) \in K. \end{cases} \tag{1}$$

We denote the set of all solutions $\left(x^*, y^*, z^*, u_1^*, v_1^*, u_2^*, v_2^*, u_3^*, v_3^*\right)$ of problem (1) by $\mathrm{SNSVVID}(\Xi, \mathfrak{A}, \mathfrak{B}, \wedge, K)$.

We first recall some basic concepts and well known results.

Definition 1. *A mapping $g : H \to H$ is said to be*

(i) monotone, if

$$\langle g(x) - g(y), x - y \rangle \geq 0 \ \ \forall x, y \in H;$$

(ii) strictly monotone, if g is monotone and

$$\langle g(x) - g(y), x - y \rangle = 0 \ \ \textit{if and only if } x = y;$$

(iii) υ-strongly monotone, if there exists a constant $\upsilon > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \upsilon \|x - y\|^2, \ \ \forall x, y \in H;$$

(iv) Lipschitz continuous, if there exists a constant $\upsilon > 0$ such that

$$\|g(x) - g(y)\| \leq \upsilon \|x - y\|, \ \forall x, y \in H.$$

Definition 2. *A set valued mapping $A : H \to 2^H$ is said to be υ-strongly monotone, if there exists a constant $\upsilon > 0$ such that*

$$\langle w_1 - w_2, x - y \rangle \geq \upsilon \|x - y\|^2, \quad \forall x, y \in H, w_1 \in A(x), w_2 \in Ay.$$

Definition 3. *A set valued mapping $A : H \to CB(H)$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that*

$$\mathcal{H}(Ax, Ay) \leq \tau \|x - y\|, \ \ \forall x, y \in H,$$

where $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(\mathcal{H})$.

Definition 4. *(Brezis 1973)*

If M is maximal monotone operator on H then for any $\lambda > 0$ the resolvent operator associated with M is defined by

$$J_M(x) = (I + \lambda M)^{-1}(x), \ \forall x \in H.$$

It is well know that a monotone operator is maximal iff its resolvent operator is defined every where. Furthermore the resolvent operator is single valued and nonexpansive. In particular the subdifferential $\partial\phi$ of a proper convex lower semicontinuous function $\phi : H \to (-\infty, +\infty)$ is a maximal monotone operator.

Lemma 1. *(Brezis 1973) The points $u, z \in H$ satisfies the inequality*

$$\langle u - z, x - u\rangle + \lambda\phi(x) - \lambda\phi(u) \geq 0, \ \ \forall x \in H,$$

if and only if

$$u = J_\phi^\lambda(z),$$

where $J_\phi^\lambda = (I + \lambda\partial\phi)^{-1}$ is a resolvent operator and $\lambda > 0$ is a constant.

For any $x, y \in H$, J_ϕ^λ is nonexpansive, i.e.,

$$\|J_\phi^\lambda(x) - J_\phi^\lambda(y)\| \leq \|x - y\|, \ \forall x, y \in H.$$

Assume that $g : H \to H$ is a surjective mapping and from Lemma 1 and (1) we have the following proximal point problem:

$$\begin{cases} g_1(x^*) &= J_\phi^{r_1}\left[g_1(y^*) - r_1 N_1\left(u_1^*, v_1^*\right)\right], \\ g_2(y^*) &= J_\phi^{r_2}\left[g_2(z^*) - r_2 N_2\left(u_2^*, v_2^*\right)\right], \\ g_3(z^*) &= J_\phi^{r_3}\left[g_3(x^*) - r_3 N_3\left(u_3^*, v_3^*\right)\right], \end{cases} \tag{2}$$

provided $K \subset g_i(H)$ for each $i = 1, 2, 3$.

Lemma 2. *(Weng 1991)*

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative real numbers such that

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n \ \ \forall n > n_0,$$

where n_0 is a nonnegative integer, $\{t_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} b_n = 0(t_n)$ and $\sum_{n=0}^{\infty} c_n < +\infty$. Than $a_n \to 0$ as $n \to +\infty$.

Definition 5. *Let $A, B : H \to 2^H$ be set valued mappings and $N : H \times H \to H$ be a nonlinear mapping.*

(i) *N is said to be A-strongly monotone with respect to the first argument, if there exists a constant $\upsilon > 0$ such that for all $x, y \in H$*

$$\left\langle N(u_1, w) - N(u_2, w), x - y\right\rangle \geq \upsilon\|x - y\|^2 \ \forall u_1 \in A(x), u_2 \in A(y), \ w \in H;$$

(ii) *N is said to be B-relaxed monotone with respect to the second argument, if there exists a constant $\xi > 0$ such that for all $x, y \in H, v_1 \in B(x), v_2 \in B(y)$*

$$\left\langle N(u, v_1) - N(u, v_2), x - y\right\rangle \geq -\xi\|x - y\|^2, \ \forall u \in H.$$

Main results

We begin with some observations which are related to the problem (1).

Remark 1. *If $(x^*, y^*, z^*) \in SNSVVID(\Xi, \mathfrak{A}, \mathfrak{B}, \wedge, K)$, by (2) we have that*

$$x^* = x^* - g_1(x^*) + J_\phi^{r_1}\left[g_1(y^*) - r_1 N_1\left(u_1^*, v_1^*\right)\right]. \tag{3}$$

provided $K \subset g_1(H)$.

Consequently if S is a Lipschitz mapping such that $x^* \in F(S)$, then it follows from (3) that

$$x^* = S(x^*) = S\left(x^* - g_1(x^*) + J_\phi^{r_1}\left[g_1(y^*) - r_1 N_1\left(u_1^*, v_1^*\right)\right]\right). \tag{4}$$

By virtue of (4) and Nadler's Theorem (Nadler 1969), we suggest the following iterative algorithm.

Algorithm 1 Let ϵ_n be a sequence of nonnegative real number with $\epsilon_n \to 0$ as $n \to \infty$. Let r_1, r_2, r_3 be three given positive real numbers in $(0, 1)$. For arbitrary chosen initial $x_0 \in H$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in H, such that

$$\begin{cases} g_3(z_n) &= J_\phi^{r_3}\left[g_3(x_n) - r_3 N_3(u_{n,3}, v_{n,3})\right], \\ g_2(y_n) &= J_\phi^{r_2}\left[g_2(z_n) - r_2 N_2(u_{n,2}, v_{n,2})\right], \quad \forall n \geq 1 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S\left(x_n - g_1(x_n) + J_\phi^{r_1}\left[g_1(y_n) - r_1 N_1(u_{n,1}, v_{n,1})\right]\right), \end{cases} \tag{5}$$

where

$$\begin{cases} u_{n,3} \in A_3(x_n), u_{n-1,3} \in A_3(x_{n-1}) : \|u_{n,3} - u_{n-1,3}\| \leq (1+\epsilon_n)\mathcal{H}(A_3(x_n), A_3(x_{n-1})), \\ v_{n,3} \in B_3(x_n), v_{n-1,3} \in B_3(x_{n-1}) : \|v_{n,3} - v_{n-1,3}\| \leq (1+\epsilon_n)\mathcal{H}(B_3(x_n), B_3(x_{n-1})), \\ u_{n,2} \in A_2(z_n), u_{n-1,2} \in A_2(z_{n-1}) : \|u_{n,2} - u_{n-1,2}\| \leq (1+\epsilon_n)\mathcal{H}(A_2(z_n), A_2(z_{n-1})), \\ v_{n,2} \in B_2(z_n), v_{n-1,2} \in B_2(z_{n-1}) : \|v_{n,2} - v_{n-1,2}\| \leq (1+\epsilon_n)\mathcal{H}(B_2(z_n), B_2(z_{n-1})), \\ u_{n,1} \in A_1(y_n), u_{n-1,1} \in A_1(y_{n-1}) : \|u_{n,1} - u_{n-1,1}\| \leq (1+\epsilon_n)\mathcal{H}(A_1(y_n), A_1(y_{n-1})), \\ v_{n,1} \in B_1(y_n), v_{n-1,1} \in B_1(y_{n-1}) : \|v_{n,1} - v_{n-1,1}\| \leq (1+\epsilon_n)\mathcal{H}(B_1(y_n), B_1(y_{n-1})), \end{cases} \tag{6}$$

and $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $S : H \to H$ is a mapping.

Theorem 1. *Let K be a nonempty closed and convex subset of a real Hilbert space H and $\phi : H \to (-\infty, +\infty)$ be a proper convex lower semicontinuous function. Let $A_i : H \to 2^H$ be a μ_i-Lipschitz continuous mapping with $\mu_i < 1$ and $B_i : H \to 2^H$ be a σ_i-Lipschitz continuous mapping with $\sigma_i < 1$, $i = 1, 2, 3$. Let $N_i : H \times H \to H$ be a ρ_i-Lipschitz continuous with respect to the first variable and η_i-Lipschitz continuous with respect to the second variable and N_i be A_i-strongly monotone with constant $\upsilon_i > 0$ and B_i-relaxed monotone with constant $\xi_i > 0$, $i = 1, 2, 3$. Let $g_i : H \to H$ be a λ_i-strongly monotone and γ_i-Lipschitz continuous mapping, $i = 1, 2, 3$. Let $S : H \to H$ be a τ-Lipschitz continuous mapping with $0 < \tau \leq 1$. If $SNSVVID(\Xi, \mathfrak{A}, \mathfrak{B}, \wedge, K) \cap F(S) \neq \emptyset$, and the following conditions are satisfied:*

(i)

$$h_i \in \left[0,\ \frac{(\rho_i\mu_i + \eta_i\sigma_i) - \sqrt{(\rho_i\mu_i + \eta_i\sigma_i)^2 - (\upsilon_i - \xi_i)^2}}{2(\rho_i\mu_i + \eta_i\sigma_i)}\right) \bigcup \left[\frac{(\rho_i\mu_i + \eta_i\sigma_i) + \sqrt{(\rho_i\mu_i + \eta_i\sigma_i)^2 - (\upsilon_i - \xi_i)^2}}{2(\rho_i\mu_i + \eta_i\sigma_i)}, 1\right)$$

where $h_i = \sqrt{1 - 2\lambda_i + \gamma_i^2}$, $i = 1, 2, 3$;

(ii)

$$|r_i - \frac{\upsilon_i-\xi_i}{(\rho_i\mu_i+\eta_i\sigma_i)^2}| < \frac{\sqrt{(\upsilon_i-\xi_i)^2-(\rho_i\mu_i+\eta_i\sigma_i)^2(4h_i)(1-h_i)}}{(\rho_i\mu_i+\eta_i\sigma_i)^2}, \quad i=1,2,3;$$

(iii) for each $i = 1,2,3$

$$\frac{\Phi_{n,N_i}(r_i)+h_i}{1-h_i} \leq \frac{\Phi_{N_i}(r_i)+h_i}{1-h_i} < 1,$$

where

$$\begin{cases} \Phi_{N_i}(r_i) = \sqrt{1-2r_i(\upsilon_i-\xi_i)+r_i^2((\rho_i\mu_i+\eta_i\sigma_i)(1+M))^2}; \\ \Phi_{n,N_i}(r_i) = \sqrt{1-2r_i(\upsilon_i-\xi_i)+r_i^2((\rho_i\mu_i+\eta_i\sigma_i)(1+\epsilon_n))^2}; \end{cases} \tag{7}$$

where $M = \sup_{n\geq 1}\epsilon_n$.

(iv) $\{\alpha_n\} \subset (0,1)$ *such that* $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Then the sequences $\{x_n\},\{y_n\},\{z_n\},\{u_{n,i}\},\{v_{n,i}\}$ *suggested by Algorithm 1 converge strongly to* x^*,y^*,z^*, u_i^*,v_i^* $i = 1,2,3$ *respectively, and* $(x^*,y^*,z^*,u_i^*,v_i^*) \in SNSVVID(\Xi,\mathfrak{A},\mathfrak{B},\wedge,K)$, $x^* \in F(S)$.

Proof. Let $\left(x^*,y^*,z^*,u_i^*,v_i^*\right) \in SNSVVID(\Xi,\mathfrak{A},\mathfrak{B},\wedge,K)$ and $x^* \in F(S)$. By (2) and (4) we have

$$\begin{cases} g_3(z^*) = J_\phi^{r_3}[g_3(x^*) - r_3N_3\left(u_3^*,v_3^*\right)], \\ g_2(y^*) = J_\phi^{r_2}[g_2(z^*) - r_2N_2\left(u_2^*,v_2^*\right)], \\ x^* = (1-\alpha_n)x^* + \alpha_n S\left(x^* - g_1(x^*) + J_\phi^{r_1}\left[g_1(y^*) - r_1N_1\left(u_1^*,v_1^*\right)\right]\right) \end{cases} \tag{8}$$

Consequently, by (5) and (6), we have

$$\begin{aligned} &\|x_{n+1}-x^*\| \\ &= \|(1-\alpha_n)x_n + \alpha_n S\left(x_n - g_1(x_n) + J_\phi^{r_1}\left[g_1(y_n) - r_1N_1(n_{n,1},v_{n,1})\right]\right) - x^*\| \\ &\leq (1-\alpha_n)\|x_n-x^*\| + \alpha_n\|S\left(x_n - g_1(x_n) + J_\phi^{r_1}\left[g_1(y_n) - r_1N_1(u_{n,1},v_{n,1})\right]\right) \\ &\quad - S\left(x^* - g_1(x^*) + J_\phi^{r_1}\left[g_1(y^*) - r_1N_1\left(u_1^*,v_1^*\right)\right]\right)\| \\ &\leq (1-\alpha_n)\|x_n-x^*\| + \alpha_n\tau\Big[\|x_n - x^* - (g_1(x_n)-g_1(x^*))\| \\ &\quad + \|J_\phi^{r_1}[g_1(y_n)-r_1N_1(u_{n,1},v_{n,1})] - J_\phi^{r_1}\left[g_1(y^*)-r_1N_1\left(u_1^*,v_1^*\right)\right]\|\Big] \\ &\leq (1-\alpha_n)\|x_n-x^*\| + \alpha_n\tau\left[\|x_n - x^* - (g_1(x_n)-g_1(x^*))\|\right. \\ &\quad \left.+ \|y_n - y^* - (g_1(y_n)-g_1(y^*))\| + \|y_n - y^* - r_1\left(N_1(u_{n,1},v_{n,1}) - N_1\left(u_1^*,v_1^*\right)\right)\|\right]. \end{aligned} \tag{9}$$

Since $N_1(\cdot,\cdot)$ is ρ_1-Lipschitz continuous with respect to the first variable and η_1-Lipschitz continuous with respect to the second variable, and A_1 is μ_1-Lipschitz continuous, and B_1 is σ_1-Lipschitz continuous, we have

$$\begin{aligned} \|N_1(u_{n,1},v_{n,1}) - N_1\left(u_1^*,v_1^*\right)\| &\leq \rho_1\|u_{n,1}-u_1^*\| + \eta_1\|v_{n,1}-v_1^*\| \\ &\leq \rho_1(1+\epsilon_n)\mathcal{H}(A_1(y_n),A_1(y^*)) + \eta_1(1+\epsilon_n)\mathcal{H}(B_1(y_n),B_1(y^*)) \\ &\leq \rho_1\mu_1(1+\epsilon_n)\|y_n-y^*\| + \eta_1\sigma_1(1+\epsilon_n)\|y_n-y^*\| \\ &\leq (\rho_1\mu_1+\eta_1\sigma_1)(1+\epsilon_n)\|y_n-y^*\|. \end{aligned} \tag{10}$$

Since N_1 is A_1-strongly monotone with constant $\upsilon_1 > 0$ and B_1-relaxed monotone with constant $\xi_i > 0$, it follows from (10) that

$$\begin{aligned}
&\|y_n - y^* - r_1\left(N_1(u_{n,1}, v_{n,1}) - N_1\left(u_1^*, v_1^*\right)\right)\|^2\\
&= \|y_n - y^*\|^2 - 2r_1\left\langle N_1(u_{n,1}, v_{n,1}) - N_1\left(u_1^*, v_1^*\right), y_n - y^*\right\rangle\\
&\quad + r_1^2\|N_1(u_{n,1}, v_{n,1}) - N_1\left(u_1^*, v_1^*\right)\|^2\\
&= \|y_n - y^*\|^2 - 2r_1\langle N_1(u_{n,1}, v_{n,1}) - N_1(u_1^*, v_{n,1}), y_n - y^*\rangle\\
&\quad -2r_1\left\langle N_1\left(u_1^*, v_{n,1}\right) - N_1\left(u_1^*, v_1^*\right), y_n - y^*\right\rangle + r_1^2\|N_1(u_{n,1}, v_{n,1}) - N_1\left(u_1^*, v_1^*\right)\|^2\\
&\le \|y_n - y^*\|^2 - 2r_1\upsilon_1\|y_n - y^*\|^2 + 2r_1\xi_1\|y_n - y^*\|^2\\
&\quad + r_1^2((\rho_1\mu_1 + \eta_1\sigma_1)(1+\epsilon_n))^2\|y_n - y^*\|^2\\
&\le \left(1 - 2r_1\upsilon_1 + 2r_1\xi_1 + r_1^2((\rho_1\mu_1 + \eta_1\sigma_1)(1+\epsilon_n))^2\right)\|y_n - y^*\|^2
\end{aligned}$$

i.e.,

$$\|y_n - y^* - r_1(N_1(u_{n,1}, v_{n,1}) - N_1(u_1^*, v_1^*))\|^2 \le (\Phi_{nN_1}(r_1))^2\|y_n - y^*\|^2, \tag{11}$$

where

$$\Phi_{n,N_1}(r_1) := \sqrt{1 - 2r_1(\upsilon_1 - \xi_1) + r_1^2((\rho_1\mu_1 + \eta_1\sigma_1)(1+\epsilon_n))^2}.$$

Note that

$$\begin{aligned}
\|y_n - y^*\| &= \|y_n - y^* - \left[g_2(y_n) - g_2(y^*)\right] + \left[g_2(y_n) - g_2(y^*)\right]\|\\
&\le \|y_n - y^* - [g_2(y_n) - g_2(y^*)]\| + \|g_2(y_n) - g_2(y^*)\|.
\end{aligned} \tag{12}$$

Since g_2 is λ_2-strongly monotone and γ_2-Lipschitz continuous mapping, we have

$$\begin{aligned}
&\|y_n - y^* - [g_2(y_n) - g_2(y^*)]\|^2\\
&= \|y_n - y^*\|^2 - 2\langle g_2(y_n) - g_2(y^*), y_n - y^*\rangle + \|g_2(y_n) - g_2(y^*)\|^2\\
&\le \|y_n - y^*\|^2 - 2\lambda_2\|y_n - y^*\|^2 + \gamma_2^2\|y_n - y^*\|^2\\
&\le (1 - 2\lambda_2 + \gamma_2^2)\|y_n - y^*\|^2\\
&= (h_2)^2\|y_n - y^*\|^2,
\end{aligned} \tag{13}$$

where $h_2 = \sqrt{1 - 2\lambda_2 + \gamma_2^2}$.

On the other hand, by (2) and (5), we have

$$\begin{aligned}
&\|g_2(y_n) - g_2(y^*)\|\\
&= \|J_\phi^{r_2}[g_2(z_n) - r_2N_2(u_{n,2}, v_{n,2})] - J_\phi^{r_2}\left[g_2(z^*) - r_2N_2\left(u_2^*, v_2^*\right)\right]\|\\
&\le \|g_2(z_n) - g_2(z^*) - r_2(N_2(u_{n,2}, v_{n,2}) - N_2(u_2^*, v_2^*))\|\\
&\le \|z_n - z^* - (g_2(z_n) - g_2(z^*))\| + \|z_n - z^* - r_2\left(N_2(u_{n,2}, v_{n,2}) - N_2\left(u_2^*, v_2^*\right)\right)\|.
\end{aligned} \tag{14}$$

In view of the assumptions of N_2, A_2, B_2, g_2 and by using the same method as given in the proofs in (11) and (13), we can obtain that

$$\|z_n - z^* - r_2(N_2(u_{n,2}, v_{n,2}) - N_2(u_2^*, v_2^*))\|^2 \le (\Phi_{n,N_2}(r_2))^2\|z_n - z^*\|^2, \tag{15}$$

where

$$(\Phi_{n,N_2}(r_2) = \sqrt{1 - 2r_2(\upsilon_2 - \xi_2) + r_2^2((\rho_2\mu_2 + \eta_2\sigma_2)(1+\epsilon_n))^2}$$

and

$$\|z_n - z^* - (g_2(z_n) - g_2(z^*))\|^2 \le (h_2)^2\|z_n - z^*\|^2. \tag{16}$$

From (15), (16) and (14), we have

$$\|g_2(y_n) - g_2(y^*)\| \le (\Phi_{n,N_2}(r_2) + h_2)\|z_n - z^*\|. \tag{17}$$

Combining (12), (13) and (17) we obtained

$$\|y_n - y^*\| \le h_2\|y_n - y^*\| + (\Phi_{nN_2}(r_2) + h_2)\|z_n - z^*\|. \tag{18}$$

Observe that

$$\begin{aligned} \|z_n - z^*\| &= \|z_n - z^* - [g_3(z_n) - g_3(z^*)] + [g_3(z_n) - g_3(z^*)]\| \\ &\le \|z_n - z^* - [g_3(z_n) - g_3(z^*)]\| + \|g_3(z_n) - g_3(z^*)\|. \end{aligned} \tag{19}$$

and in view of (2) and (5), we have

$$\begin{aligned} \|g_3(z_n) - g_3(z^*)\| \le\ & \|x_n - x^* - [g_3(x_n) - g_3(x^*)]\| \\ & + \|x_n - x^* - r_3\left(N_3(u_{n,3}, v_{n,3}) - N_3\left(u_3^*, v_3^*\right)\right)\|. \end{aligned} \tag{20}$$

By using the assumptions on N_3, A_3, B_3 and g_3, we have

$$\|x_n - x^* - r_3\left(N_3(u_{n,3}, v_{n,3}) - N_3\left(u_3^*, v_3^*\right)\right)\|^2 \le \left(\Phi_{n,N_3}(r_3)\right)^2\|x_n - x^*\|^2. \tag{21}$$

where

$$\Phi_{n,N_3}(r_3) = \sqrt{1 - 2r_3(\upsilon_3 - \xi_3) + r_3^2((\rho_3\mu_3 + \eta_3\sigma_3)(1+\epsilon_n))^2}$$

$$\|x_n - x^* - [g_3(x_n) - g_3(x^*)]\|^2 \le (h_3)^2\|x_n - x^*\|^2. \tag{22}$$

$$\|z_n - z^* - [g_3(z_n) - g_3(z^*)]\|^2 \le (h_3)^2\|z_n - z^*\|^2. \tag{23}$$

Substituting (21) and (22) into (20), we have

$$\|g_3(z_n) - g_3(z^*)\| \le (\Phi_{n,N_3}(r_3) + h_3)\|x_n - x^*\|. \tag{24}$$

Combining (19), (23) and (24), it yields that

$$\|z_n - z^*\| \le h_3\|z_n - z^*\| + (\Phi_{n,N_3}(r_3) + h_3)\|x_n - x^*\|. \tag{25}$$

This imply that

$$\|z_n - z^*\| \le \frac{(\Phi_{n,N_3}(r_3) + h_3)}{1 - h_3}\|x_n - x^*\|. \tag{26}$$

Substituting (26) into (18) we have

$$\|y_n - y^*\| \le h_2\|y_n - y^*\| + \frac{(\Phi_{n,N_2}(r_2)+h_2)(\Phi_{n,N_3}(r_3)+h_3)}{1-h_3}\|x_n - x^*\|, \tag{27}$$

that is

$$\|y_n - y^*\| \le \frac{(\Phi_{n,N_2}(r_2) + h_2)(\Phi_{n,N_3}(r_3) + h_3)}{(1-h_2)(1-h_3)}\|x_n - x^*\|. \tag{28}$$

From (11) and (28), we get

$$\begin{aligned}&\|y_n - y^* - r_1[N_1(u_{n,1}, v_{n,1}) - N_1(u_1^*, v_1^*)]\| \\ &\le \frac{(\Phi_{n,N_1}(r_1))(\Phi_{n,N_2}(r_2) + h_2)(\Phi_{n,N_3}(r_3) + h_3)}{(1-h_2)(1-h_3)}\|x_n - x^*\|. \end{aligned} \tag{29}$$

On the other hand, since g_1 is λ_1-strongly monotone and γ_1-Lipschitz continuous mapping, we have

$$\begin{aligned}\|x_n - x^* - (g_1(x_n) - g_1(x^*))\|^2 &= ||x_n - x^*||^2 + ||g_1(x_n) - g_1(x^*)||^2 \\ &\quad - 2\langle x_n - x^*, g_1(x_n) - g_1(x^*)\rangle \\ &\le (1 - 2\lambda_1 + \gamma_1^2)||x_n - x^*||^2 = h_1^2||x_n - x^*||^2,\end{aligned}$$

i.e.,

$$\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \le h_1\|x_n - x^*\|. \tag{30}$$

Similarly, we have

$$\|y_n - y^* - (g_1(y_n) - g_1(y^*))\| \le h_1\|y_n - y^*\|. \tag{31}$$

Substituting (28) into (31), we have

$$\begin{aligned}&\|y_n - y^* - (g_1(y_n) - g_1(y^*))\| \\ &\le h_1\frac{(\Phi_{n,N_2}(r_2) + h_2)(\Phi_{n,N_3}(r_3) + h_3)}{(1-h_2)(1-h_3)}\|x_n - x^*\|.\end{aligned} \tag{32}$$

Set

$$\ell_n = \frac{(\Phi_{n,N_2}(r_2) + h_2)(\Phi_{n,N_3}(r_3) + h_3)}{(1-h_2)(1-h_3)}. \tag{33}$$

Substituting (30), (31), (32) and (33) into (9), we get

$$\|x_{n+1} - x^*\| \le (1 - \alpha_n(1 - \tau(h_1 + h_1\ell_n + \Phi_{n,N_1}(r_1)\ell_n)))\|x_n - x^*\|. \tag{34}$$

Since

$$\begin{aligned}\Phi_{n,N_i}(r_i) &:= \sqrt{1 - 2r_i(\upsilon_i - \xi_n) + r_i^2((\rho_i\mu_i + \eta_i\sigma_i)(1+\epsilon_n))^2} \\ &\le \sqrt{1 - 2r_i(\upsilon_i - \xi_n) + r_i^2((\rho_i\mu_i + \eta_i\sigma_i)(1+M))^2} := \Phi_{N_i}(r_i),\end{aligned}$$

letting $\ell := \frac{(\Phi_{N_2}(r_2)+h_2)(\Phi_{N_3}(r_3)+h_3)}{(1-h_2)(1-h_3)}$, then we have $\ell_n \le \ell$. Therefore from (34) we have that

$$\| x_{n+1} - x^*\| \le (1 - \alpha_n(1 - \tau(h_1 + h_1\ell + \Phi_{N_1}(r_1)\ell)))\|x_n - x^*\|. \tag{35}$$

By condition (iii)

$$\prod_{i=1}^{3}\frac{\Phi_{N_i}(r_i) + h_i}{1 - h_i} < 1, \tag{36}$$

this imply that

$$\ell < \frac{1 - h_1}{\Phi_{N_1}(r_1) + h_1} \tag{37}$$

that is

$$\Im := h_1 + h_1\ell + \Phi_{N_1}(r_1)\ell < 1. \tag{38}$$

Put

$$\begin{cases} a_n &= \|x_n - x^*\| \\ t_n &= \alpha_n(1 - \tau\Im). \end{cases} \tag{39}$$

By the assumption that $0 < \tau \leq 1$, it follows that

$$\tau\Im \in (0, 1).$$

This imply that $t_n \in (0, 1)$. From assumption (iv) we have

$$\sum_{n=0}^{\infty} t_n = \infty.$$

These show that all conditions in Lemma 2 are satisfied. Hence $x_n \to x^*$ as $n \to \infty$. Consequently from (26) and (28), we have $z_n \to z^*$ and $y_n \to y^*$ as $n \to \infty$, respectively. Moreover since A_i is μ_i-Lipschitz continuous and B_i is σ_i-Lipschitz continuous with $\mu_i < 1,\ \sigma_i < 1$, we can also prove that $\{u_{n,i}\}$ and $\{v_{n,i}\},\ i = 1, 2, 3$ are Cauchy sequences. Thus there exists $u_i^*, v_i^* \in H$ such that $u_{n,i} \to u_i^*, v_{n,i} \to v_i^*,\ (i = 1, 2, 3)$ as $n \to \infty$. Moreover by using the continuity of mappings $A_i, B_i, g_i, N_i, J_\phi^{r_i},\ i = 1, 2, 3$, it follows from (5) that

$$g_3(z^*) = J_\phi^{r_3}\left[g_3(x^*) - r_3N_3\left(u_3^*, v_3^*\right)\right],$$

$$g_2(y^*) = J_\phi^{r_2}\left[g_2(z^*) - r_2N_2\left(u_2^*, v_2^*\right)\right],$$

$$x^* = S\left(x^* - g_1(x^*) + J_\phi^{r_1}\left[g_1(y^*) - r_1N_1\left(u_1^*, v_1^*\right)\right]\right).$$

Hence from Lemma 2 it follows that $\left(x^*, y^*, z^*, u_i^*, v_i^*\right) \in \mathrm{SNSVVID}(\Xi, \mathfrak{A}, \mathfrak{B}, \wedge, K)$. Finally we prove that $u_i^* \in A_i(y^*)$ and $v_i^* \in B_1(y^*)$ Indeed we have

$$\begin{aligned} d(u_1^*, A_1(y^*)) &= \inf\{\|u_1^* - w\| : w \in A_1(y^*)\} \\ &\leq \|u_1^* - u_{n,1}\| + d(u_{n,1}, A_1(y^*)) \\ &\leq \|u_1^* - u_{n,1}\| + \mathcal{H}(A_1(y_n), A_1(y^*)) \\ &\leq \|u_1^* - u_{n,1}\| + \mu_1\|y_n - y^*\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

That is $d(u_1^*, A_1(y^*)) = 0$. Since $A_1(y^*) \in CB(H)$, we must have $u_1^* \in A_1(y^*)$. Similarly we can show that $u_2^* \in A_2(z^*), u_3^* \in A_3(x^*), v_1^* \in B_1(y^*), v_2^* \in B_2(z^*)$ and $v_3^* \in B_3(x^*)$. This complete the proof. ■

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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