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Hierarchical fixed points of strictly pseudo contractive mappings for variational inequality problems

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Abstract

In this paper, we introduce a new iterative scheme that converges strongly to a common fixed point of a countable family of strictly pseudo-contractive mappings in a real Hilbert space which is also a solution of variational inequality problem related to quadratic minimization problems. Our results extend ones of Yao et al. [Math. and Comput. Modell. 52(9-10):1697–1705, 2010], Gu et al. [J. Appl. Math. 2011:17 p., 2011] and some authors.

Keywords: Strong convergence; Strictly pseudo-contractive mapping; Hilbert spaces; Variational inequality; Hierarchical fixed point problem

Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively and let C is a nonempty closed and convex subset of H . A mapping $f : C \rightarrow H$ is called a *contraction* on C if there exists a constant $\rho \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \rightarrow H$ is called a *nonexpansive* on C if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping $M : H \rightarrow H$ is called a *strongly monotone operator* with coefficient α if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Mx - My \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in H$$

and M is called a *monotone operator* if

$$\langle x - y, Mx - My \rangle \geq 0, \quad \forall x, y \in H.$$

It is well known that the mapping $(I - T)$ is a monotone operator, if T is a nonexpansive mapping and I is a identity mapping.

A mapping $T : C \rightarrow H$ is called *k-strict pseudo-contraction* if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

If there exists a point $x \in C$ such that $x = Tx$, then x is said to be fixed point of T . We denote the set of all fixed points of T by $F(T)$. It is well known that $F(T)$ is closed and convex if T is nonexpansive.

Note that the class of *k-strict pseudo-contraction* mappings includes the class of nonexpansive mappings on C as a subclass. i.e., T is nonexpansive if and only if T is *0-strict pseudo-contraction*. Recently, many authors have been devoting the studies on the problems of finding fixed points for *k-strict pseudo-contraction* mappings; see Acedo and Xu (2007); Cho et al. (2009); Jung (2010); Jung (2011); Zhou (2008) and the references therein.

A variational inequality in a real Hilbert space H is formulated as finding a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \quad (1)$$

where $F : C \rightarrow H$ is a nonlinear mapping. We denote the set of solution of (1) by $VI(C, F)$. If F is a monotone operator, then (1) is also known as a *monotone variational inequality*.

For given nonlinear operators F, g , we consider the problem of finding $u \in H$ such that

$$\langle F(u), g(v) - g(u) \rangle \geq 0, \quad \forall g(v), g(u) \in C \quad (2)$$

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which is known as *the general variational inequality problem*. For given nonlinear operators F, g, h , we consider the problem of finding $u \in H : h(u) \in C$ such that

$$\langle F(u), g(v) - h(u) \rangle \geq 0, \quad \forall g(v) \in C \quad (3)$$

which is called *the extended general variational inequality*.

The variational inequalities have been studied widely and are being used as a mathematical programming tool in modeling a wide class of problems arising in several branches of pure and applied sciences; see Baiocchi and Capelo (1984); Giannessi and Maugeri (1995); Kinderlehrer and Stampacchia (1980). For general variational inequalities and extended general variational inequalities, we can refer Noor (2004,2009); Noor et al. (2012a,2012b) and references therein.

It is well-known that the variational inequality (1) is equivalent to the fixed point equation

$$x^* = P_C[(I - \gamma F)x^*], \quad (4)$$

where $\gamma > 0$ and P_C is the metric projection of H onto C which assigns, to each $x \in H$, the unique point in C , denoted $P_C[x]$, such that

$$\|x - P_C[x]\| = \inf\{\|x - y\| : y \in C\}.$$

Therefore, fixed point algorithms can be applied to solve variational inequalities.

The following problem is called a *hierarchical fixed point problem*: Find $x^* \in F(T)$ such that

$$\langle x^* - Sx^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \quad (5)$$

where $S : C \rightarrow H$ be a mapping. It is known that the hierarchical fixed point problem (5) links with some monotone variational inequalities and convex programming problems; see Gu et al. (2011); Yao et al. (2010).

In order to solve the hierarchical fixed point problem (5), Moudafi (2007) introduced the following Krasnoselski-Mann algorithm:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\beta_n Sx_n + (1 - \beta_n)Tx_n), \quad (6)$$

where $S, T : C \rightarrow C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Then he showed that $\{x_n\}$ converges weakly to a fixed point of T which is a solution of problem (5). For obtaining a strong convergence result, in Mainge and Moudafi (2007) and Marino and Xu (2011) introduced the following algorithm:

$$x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n(\beta_n Sx_n + (1 - \beta_n)Tx_n), \quad (7)$$

where $f : C \rightarrow C$ is a contraction mapping, S and $T : C \rightarrow C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Then they showed that $\{x_n\}$ converges strongly to a fixed point of T which is a solution of problem (5).

On the other hand, Cianciaruso et al. (2009) introduced a two step algorithm to solve the problem (5) as follows:

$$\begin{cases} y_n = \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Ty_n, \end{cases} \quad (8)$$

where $f : C \rightarrow C$ is a contraction mapping, S and $T : C \rightarrow C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Under some certain restrictions on parameters, the authors proved the sequence $\{x_n\}$ generated by (8) converges strongly to $x^* \in F(T)$, which is a unique solution of the following variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \quad (9)$$

By changing the restrictions on parameters, the authors obtained another result on the iterative scheme (8), i.e., the sequence $\{x_n\}$ generated by (8) converges strongly to $x^* \in F(T)$, which is a unique solution of the following variational inequality:

$$\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T), \quad (10)$$

where $\tau \in (0, \infty)$ is a constant.

In 2010, Yao et al. (2010) modified the two step algorithm (8) to extend Range of f from C to H by using the metric projection of H onto C . They introduced the following iterative scheme:

$$\begin{cases} y_n = \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1 - \alpha_n)Ty_n], \end{cases} \quad (11)$$

where $f : C \rightarrow H$ is a contraction mapping, S and $T : C \rightarrow C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. The authors proved the sequence $\{x_n\}$ generated by (11) converges strongly to $x^* \in F(T)$, which is a unique solution of one of the variational inequalities (9) and (10).

In 2011, Gu et al. (2011) introduced the following iterative algorithm:

$$\begin{cases} y_n = P_C[\beta_n Sx_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \quad (12)$$

where $f : C \rightarrow H$ is a contraction mapping, $S : C \rightarrow H$ is a nonexpansive mapping, $\{T_i\}_{i=1}^\infty : C \rightarrow C$ is a countable family of nonexpansive mappings, $\alpha_0 = 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. The authors proved the sequence $\{x_n\}$ generated by (12) converges strongly to $x^* \in F(T)$, which is a unique solution of one of the variational inequalities (9) and (10).

In this paper, motivated and inspired by the results of Gu et al. (2011), we introduce and study the following iterative scheme:

$$\begin{cases} y_n = P_C[\beta_n Sx_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)V_i y_n], \quad \forall n \geq 1, \end{cases} \quad (13)$$

where $V_i = k_i I + (1 - k_i)T_i$ and $\{T_i\}_{i=1}^\infty : C \rightarrow C$ is a countable family of k_i -strict pseudo-contraction mappings. Under some certain condition on parameters, we first prove that the sequence $\{x_n\}$ generated by (13) converges strongly to $x^* \in \bigcap_{i=1}^\infty F(T_i)$ which is a unique solution of the following variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^\infty F(T_i). \quad (14)$$

By changing the restrictions on parameters, we also prove that the sequence $\{x_n\}$ generated by (13) converges strongly to $x^* \in \bigcap_{i=1}^\infty F(T_i)$, which is a unique solution of the following variational inequality:

$$\frac{1}{\tau} \langle (I - f)x^* + (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^\infty F(T_i), \quad (15)$$

where $\tau \in (0, \infty)$ is a constant. It is easy to see that, if $k_i = 0$ for each $i \geq 1$, then our algorithm (13) is reduced to algorithm (12) of Gu et al. Also our results extend the corresponding one of Yao et al. (2010); Xu (2004); Cianciaruso et al. (2009); Moudafi (2000) and Gu et al. (2011) from the countable family of nonexpansive mappings to more general the countable family of strictly pseudo contraction mappings.

Preliminaries

This section collects some lemma which be use in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

We will use the following notation:

- (i) \rightarrow for strong convergence and \rightharpoonup for weak convergence.
- (ii) $\omega_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$ denotes the weak ω -limit set of $\{x_n\}$.

Lemma 1. Browder (1976) *Let H be a Hilbert space, C is a closed convex subset of H and $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y , then $(I - T)x = y$; in particular, if $y = 0$ then $x \in F(T)$.*

Lemma 2. Acedo and Xu (2007) *Let C be a nonempty closed convex subset of a real Hilbert space H . If $T : C \rightarrow C$ is a k -strict pseudo-contraction, then the mapping $I - T$ is demiclosed at 0. That is, if $\{x_n\}$ is a sequence in C weakly*

converging to x and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$.

Lemma 3. *Let $x \in H$ and $z \in C$ be any points. Then we have the following:*

1. *That $z = P_C[x]$ if and only if there holds the relation:*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C.$$
2. *That $z = P_C[x]$ if and only if there holds the relation:*

$$\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C.$$
3. *There holds the relation:*

$$\langle P_C[x] - P_C[y], x - y \rangle \geq \|P_C[x] - P_C[y]\|^2, \quad \forall x, y \in H.$$

Consequently, P_C is nonexpansive and monotone.

Lemma 4. Marino and Xu (2006) *Let H be a Hilbert space, C be a closed convex subset of H , $f : C \rightarrow H$ be a contraction with coefficient $0 < \rho < 1$ and $T : C \rightarrow C$ be a nonexpansive mapping. Then, for $0 < \gamma < \bar{\gamma}/\rho$, for $x, y \in C$,*

1. *the mapping $(I - f)$ is strongly monotone with coefficient $(1 - \rho)$ that is*

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \rho)\|x - y\|^2.$$
2. *the mapping $(I - T)$ is monotone, that is*

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq 0.$$

Lemma 5. Xu (2002) *Assume that $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in \mathcal{R} such that

1. $\sum_{n=1}^\infty \gamma_n = \infty$,
2. $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^\infty |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 6. Acedo and Xu (2007) *Let C be a closed convex subset of H . Let $\{x_n\}$ be a bounded sequence in H . Assume that*

- (1) *The weak ω -limit set $\omega_w(x_n) \subset C$,*
- (2) *For each $z \in C$, $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists.*

Then $\{x_n\}$ is weakly convergent to a point in C .

Lemma 7. Zhou (2008) *Let H be a real Hilbert space, C be a closed and convex subset of H , and T be a k -strict pseudo-contraction mapping on C , then $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.*

Lemma 8. Zhou (2008) Let H be a Hilbert space, C be a closed and convex subset of H , and $T : C \rightarrow H$ be a k -strict pseudo-contraction mapping. Define a mapping $V : C \rightarrow H$ by $Vx = \lambda x + (1 - \lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, V is a nonexpansive mapping such that $F(V) = F(T)$.

Lemma 9. Gu et al. (2011) Let H be a Hilbert space and C be a nonempty closed and convex subset of H . Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Then $\|Tx - x\|^2 \leq 2\langle x - Tx, x - x' \rangle, \forall x' \in F(T), \forall x \in C$.

Main results

Let us consider the net iterative scheme as follows:

$$\begin{cases} y_n = P_C[\beta_n Sx_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i y_n], \quad \forall n \geq 1, \end{cases} \tag{16}$$

where $V_i = k_i I + (1 - k_i)T_i, f : C \rightarrow H$ is a ρ -contraction mapping, $S : C \rightarrow H$ is a nonexpansive mapping, $\{T_i\}_{i=1}^\infty : C \rightarrow C$ is a countable family of k_i -strict pseudo-contraction mappings and $\cap_{i=1}^\infty F(T_i) \neq \emptyset$. Set $\alpha_0 = 1, \{\alpha_n\} \subset (0, 1)$ is a strictly decreasing sequence and $\{\beta_n\} \subset (0, 1)$. As we will see the convergence of the scheme depends on the choice of the parameters $\{\alpha_n\}$ and $\{\beta_n\}$. We list some possible hypotheses on them:

- (H1) there exists $\gamma > 0$ such that $\beta_n \leq \gamma \alpha_n$;
- (H2) $\lim_{n \rightarrow \infty} \beta_n / \alpha_n = \tau \in [0, \infty)$;
- (H3) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;
- (H4) $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
- (H5) $\sum_{n=1}^\infty |\beta_n - \beta_{n-1}| < \infty$;
- (H6) $\lim_{n \rightarrow \infty} |\alpha_n - \alpha_{n-1}| / \alpha_n = 0$;
- (H7) $\lim_{n \rightarrow \infty} |\beta_n - \beta_{n-1}| / \beta_n = 0$;
- (H8) $\lim_{n \rightarrow \infty} [|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|] / \alpha_n \beta_n = 0$;
- (H9) there exists a constant $K > 0$ such that $\frac{1}{\alpha_n} |\frac{1}{\beta_n} - \frac{1}{\beta_{n+1}}| \leq K$.

Proposition 1. Assume that (H1) holds. Then $\{x_n\}$ and $\{y_n\}$ are bounded.

Proof. Let $z \in \cap_{i=1}^\infty F(T_i) = \cap_{i=1}^\infty F(V_i)$. Then we have

$$\begin{aligned} \|x_{n+1} - z\| &= \left\| P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i y_n] - P_C[z] \right\| \\ &\leq \left\| \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i y_n - z \right\| \\ &= \left\| \alpha_n (f(x_n) - z) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (V_i y_n - z) \right\| \end{aligned}$$

$$\begin{aligned} &\leq \alpha_n \|f(x_n) - z\| + \alpha_n \|f(z) - z\| \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|V_i y_n - z\| \\ &\leq \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|y_n - z\| \\ &\leq \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|\beta_n Sx_n + (1 - \beta_n)x_n - z\| \\ &\leq \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| + (1 - \beta_n) \|x_n - z\| \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (\beta_n \|Sx_n - Sz\| + \beta_n \|Sz - z\|) \\ &\leq \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| + (1 - \beta_n) \|x_n - z\| \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (\beta_n \|x_n - z\| + \beta_n \|Sz - z\|) \\ &= \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (\|x_n - z\| + \beta_n \|Sz - z\|) \\ &= \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| \\ &\quad + (1 - \alpha_n) (\|x_n - z\| + \beta_n \|Sz - z\|) \\ &= (1 - \alpha_n (1 - \rho)) \|x_n - z\| + \alpha_n \|f(z) - z\| \\ &\quad + (1 - \alpha_n) \beta_n \|Sz - z\| \\ &\leq (1 - \alpha_n (1 - \rho)) \|x_n - z\| + \alpha_n \|f(z) - z\| \\ &\quad + \beta_n \|Sz - z\| \\ &\leq (1 - \alpha_n (1 - \rho)) \|x_n - z\| + \alpha_n [\|f(z) - z\| \\ &\quad + \gamma \|Sz - z\|]. \end{aligned} \tag{17}$$

So, by induction, one can obtain that

$$\|x_n - z\| \leq \max \left\{ \|x_0 - z\|, \frac{1}{1 - \rho} [\|f(z) - z\| + \gamma \|Sz - z\|] \right\}. \tag{18}$$

Hence $\{x_n\}$ is bounded. Of course $\{y_n\}$ is bounded too. \square

Proposition 2. Suppose that (H1) and (H3) hold. Also, assume that either (H4) and (H5) hold, or (H6) and (H7) hold. Then

(1) $\{x_n\}$ is asymptotically regular, that is,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0, \tag{19}$$

(2) the weak cluster points set $\omega_w(x_n) \subset \cap_{i=1}^{\infty} F(T_i)$.

Proof. Set $u_n = \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i y_n$. From (16) and since P_C is a nonexpansive mapping, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|P_C[u_n] - P_C[u_{n-1}]\| \\ &\leq \|u_n - u_{n-1}\| \tag{20} \\ &= \|\alpha_n(f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1})f(x_{n-1}) \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)(V_i y_n - V_i y_{n-1}) \\ &\quad + (\alpha_{n-1} - \alpha_n)V_n y_{n-1}\| \leq \alpha_n \|f(x_n) - f(x_{n-1})\| \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|y_n - y_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| (\|f(x_{n-1})\| + \|V_n y_{n-1}\|) \\ &\leq \alpha_n \rho \|x_n - x_{n-1}\| + (1 - \alpha_n) \|y_n - y_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| (\|f(x_{n-1})\| + \|V_n y_{n-1}\|). \tag{21} \end{aligned}$$

By definition of y_n one obtain that

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|P_C[\beta_n Sx_n + (1 - \beta_n)x_n] \\ &\quad - P_C[\beta_{n-1} Sx_{n-1} + (1 - \beta_{n-1})x_{n-1}]\| \\ &\leq \|(\beta_n Sx_n + (1 - \beta_n)x_n) \\ &\quad - (\beta_{n-1} Sx_{n-1} + (1 - \beta_{n-1})x_{n-1})\| \\ &= \|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} \\ &\quad + (1 - \beta_{n-1})(x_n - x_{n-1}) + (\beta_{n-1} - \beta_n)x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| (\|Sx_{n-1}\| + \|x_{n-1}\|). \tag{22} \end{aligned}$$

So, substituting (22) in (21), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n \rho \|x_n - x_{n-1}\| + (1 - \alpha_n) [\|x_n - x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| (\|Sx_{n-1}\| + \|x_{n-1}\|)] \\ &\quad + |\alpha_n - \alpha_{n-1}| (\|f(x_{n-1})\| + \|V_n y_{n-1}\|) \\ &\leq (1 - (1 - \rho)\alpha_n) \|x_n - x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| (\|Sx_{n-1}\| + \|x_{n-1}\|) \\ &\quad + |\alpha_n - \alpha_{n-1}| (\|f(x_{n-1})\| + \|V_n y_{n-1}\|). \tag{23} \end{aligned}$$

By Proposition 1, we say

$$M := \max \left\{ \sup_{n \geq 1} \{\|Sx_{n-1}\| + \|x_{n-1}\|\}, \sup_{n \geq 1} \{\|f(x_{n-1})\| + \|V_n y_{n-1}\|\} \right\}.$$

So, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - (1 - \rho)\alpha_n) \|x_n - x_{n-1}\| \\ &\quad + M[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|]. \tag{24} \end{aligned}$$

So, if (H4) and (H5) hold, we obtain the asymptotic regularity by Lemma 5, if instead, (H6) and (H7) hold, from (H1), we can write

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - (1 - \rho)\alpha_n) \|x_n - x_{n-1}\| \\ &\quad + M\alpha_n \left[\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} \right] \\ &\leq (1 - (1 - \rho)\alpha_n) \|x_n - x_{n-1}\| \\ &\quad + M\alpha_n \left[\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \gamma \frac{|\beta_n - \beta_{n-1}|}{\beta_n} \right]. \tag{25} \end{aligned}$$

By Lemma 5, we obtain the asymptotics regularity.

In order to prove (2), since $V_i x_n \in C$ for each $i \geq 1$ and $\sum_{n=1}^{\infty} (\alpha_{n-1} - \alpha_n) + \alpha_n = 1$, we have

$$\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i x_n + \alpha_n p \in C, \quad \forall p \in C. \tag{26}$$

Now, fixing $p \in \cap_{i=1}^{\infty} F(V_i)$, from (16), we have

$$\begin{aligned} &= P_C[u_n] + (1 - \alpha_n)x_n - \left(\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i x_n + \alpha_n p \right) \\ &\quad + \alpha_n p - x_{n+1} \\ &= P_C[u_n] - P_C \left[\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i x_n + \alpha_n p \right] \\ &\quad + (1 - \alpha_n)(x_n - x_{n+1}) + \alpha_n(p - x_{n+1}). \end{aligned}$$

It follows that $\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle x_n - V_i x_n, x_n - z \rangle$

$$\begin{aligned}
 &= \left\langle P_C [u_n] - P_C \left[\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i x_n + \alpha_n p \right], x_n - z \right\rangle \\
 &\quad + (1 - \alpha_n) \langle x_n - x_{n+1}, x_n - z \rangle + \alpha_n \langle p - x_{n+1}, x_n - z \rangle \\
 &\leq \left\| u_n - \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i x_n + \alpha_n p \right\| \|x_n - z\| \\
 &\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - z\| + \alpha_n \|p - x_{n+1}\| \|x_n - z\| \\
 &= \left\| \alpha_n (f(x_n) - p) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (V_i y_n - V_i x_n) \right\| \|x_n - z\| \\
 &\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - z\| + \alpha_n \|p - x_{n+1}\| \|x_n - z\| \\
 &\leq \alpha_n \|f(x_n) - p\| \|x_n - z\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|y_n - x_n\| \|x_n - z\| \\
 &\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - z\| + \alpha_n \|p - x_{n+1}\| \|x_n - z\| \\
 &\leq \alpha_n \|f(x_n) - p\| \|x_n - z\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \beta_n \|Sx_n - x_n\| \|x_n - z\| \\
 &\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - z\| + \alpha_n \|p - x_{n+1}\| \|x_n - z\| \\
 &= \alpha_n \|f(x_n) - p\| \|x_n - z\| + (1 - \alpha_n) \beta_n \|Sx_n - x_n\| \|x_n - z\| \\
 &\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - z\| + \alpha_n \|p - x_{n+1}\| \|x_n - z\|.
 \end{aligned} \tag{27}$$

Now, from Lemma 9 and (27), we get $\frac{1}{2} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|x_n - V_i x_n\|^2$

$$\begin{aligned}
 &\leq \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle x_n - V_i x_n, x_n - z \rangle \\
 &\leq \alpha_n \|f(x_n) - p\| \|x_n - z\| + (1 - \alpha_n) \beta_n \|Sx_n - x_n\| \|x_n - z\| \\
 &\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - z\| + \alpha_n \|p - x_{n+1}\| \|x_n - z\|.
 \end{aligned}$$

By (H1) and (H3), it follows that $\beta_n \rightarrow 0$, as $n \rightarrow \infty$, so that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|x_n - V_i x_n\|^2 = 0. \tag{28}$$

Since $(\alpha_{i-1} - \alpha_i) \|x_n - V_i x_n\|^2 \leq \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|x_n - V_i x_n\|^2$ for each $i \geq 1$ and $\{\alpha_n\}$ is strictly decreasing, one has

$$\lim_{n \rightarrow \infty} \|x_n - V_i x_n\| = 0, \quad \forall i \geq 1. \tag{29}$$

Hence, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = \lim_{n \rightarrow \infty} \frac{\|x_n - V_i x_n\|}{(1 - k_i)} = 0, \quad \forall i \geq 1.$$

Since $\{x_n\}$ is asymptotically regular and demiclosedness principle, we obtain the proposition. \square

Corollary 1. Suppose that the hypotheses of Proposition 2 hold. Then

- (i) $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$;
- (ii) $\lim_{n \rightarrow \infty} \|x_n - V_i y_n\| = 0, \quad \forall i \geq 1$;
- (iii) $\lim_{n \rightarrow \infty} \|y_n - V_i y_n\| = 0, \quad \forall i \geq 1$.

Proof. To prove (i), we can observe that

$$\|x_n - y_n\| \leq \beta_n \|x_n - Sx_n\|.$$

Since $\beta_n \rightarrow 0$ as $n \rightarrow \infty$, we obtain (i).

To prove (ii), we observe that

$$\|y_n - V_i x_n\| \leq \|y_n - x_n\| + \|x_n - V_i x_n\|, \quad \forall i \geq 1$$

and

$$\|x_n - V_i y_n\| \leq \|x_n - y_n\| + \|y_n - V_i x_n\|, \quad \forall i \geq 1.$$

Since $\|y_n - x_n\| \rightarrow 0$ and $\|x_n - V_i x_n\| \rightarrow 0$ as $n \rightarrow \infty$, $\forall i \geq 1$, then $\|y_n - V_i x_n\| \rightarrow 0$, that is, we obtain (ii). To prove (iii), we can observe that

$$\|y_n - V_i y_n\| \leq \|x_n - y_n\| + \|x_n - V_i y_n\|, \quad \forall i \geq 1.$$

By (i) and (ii), we obtain (iii). \square

Theorem 1. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $f : C \rightarrow H$ be a ρ -contraction mapping, $S : C \rightarrow H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^\infty : C \rightarrow C$ be a countable family of k_i -strict pseudo-contraction mappings and $\mathcal{F} = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, and $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C [\beta_n Sx_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C [\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i y_n], \quad \forall n \geq 1, \end{cases} \tag{30}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $V_i = k_i I + (1 - k_i) T_i$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5), or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle (I - f)z, x - z \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{31}$$

Proof. First of all, since $P_{\mathcal{F}}f$ is a contraction. By Banach contraction principle, so there exists a unique $z \in \mathcal{F}$ such that $z = P_{\mathcal{F}}f(z)$, Moreover, from Lemma 3(1), we have

$$\langle f(z) - z, y - z \rangle \leq 0, \quad \forall y \in \mathcal{F}.$$

Since (H2) implies (H1), thus $\{x_n\}$ is bounded. Moreover, since either (H4) and (H5) or (H6) and (H7) then $\{x_n\}$ is asymptotically regular. Similarly, by Proposition 2, the weak cluster points set of x_n , that is, $\omega_w(x_n)$, is a subset of \mathcal{F} .

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle f(z) - z, x_n - z \rangle = \lim_{k \rightarrow \infty} \langle f(z) - z, x_{n_k} - z \rangle,$$

and $x_{n_k} \rightarrow x'$. So, it follows that $x' \in \mathcal{F}$. Then, we also have

$$\lim_{k \rightarrow \infty} \langle f(z) - z, x_{n_k} - z \rangle = \langle f(z) - z, x' - z \rangle \leq 0.$$

Set $u_n = \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i y_n$, we obtain

$$\|x_{n+1} - z\|^2 = \langle P_C[u_n] - u_n, P_C[u_n] - z \rangle + \langle u_n - z, x_{n+1} - z \rangle. \tag{32}$$

By Lemma 3(1), we have

$$\langle P_C[u_n] - u_n, P_C[u_n] - z \rangle \leq 0. \tag{33}$$

From (32) and (33), it follows that $\|x_{n+1} - z\|^2$

$$\begin{aligned} &\leq \langle u_n - z, x_{n+1} - z \rangle = \alpha_n \langle f(x_n) - f(z), x_{n+1} - z \rangle \\ &\quad + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \\ &\quad \times \langle V_i y_n - z, x_{n+1} - z \rangle \\ &\leq \alpha_n \rho \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\quad \times (1 - \alpha_n) \|y_n - z\| \|x_{n+1} - z\| \leq \alpha_n \rho \|x_n - z\| \\ &\quad \times \|x_{n+1} - z\| + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\quad \times (1 - \alpha_n) \|\beta_n Sx_n + (1 - \beta_n)x_n - z\| \|x_{n+1} - z\| \end{aligned}$$

$$\begin{aligned} &\leq \alpha_n \rho \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\quad \times (1 - \alpha_n) \|x_n - z\| \|x_{n+1} - z\| \\ &\quad + (1 - \alpha_n) \beta_n \|Sz - z\| \|x_{n+1} - z\| \\ &= [1 - \alpha_n(1 - \rho)] \|x_n - z\| \|x_{n+1} - z\| \\ &\quad + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\quad + (1 - \alpha_n) \beta_n \|Sz - z\| \|x_{n+1} - z\| \\ &\leq \left[\frac{1 - \alpha_n(1 - \rho)}{2} \right] \left[\|x_n - z\|^2 + \|x_{n+1} - z\|^2 \right] \\ &\quad + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle + (1 - \alpha_n) \beta_n \|Sz - z\| \\ &\quad \times \|x_{n+1} - z\| \leq \left[1 - \frac{2(1 - \rho)\alpha_n}{1 + (1 - \rho)\alpha_n} \right] \|x_n - z\|^2 \\ &\quad + \left[\frac{2\alpha_n}{1 + (1 - \rho)\alpha_n} \right] \cdot \langle f(z) - z, x_{n+1} - z \rangle \\ &\quad + \left[\frac{2(1 - \alpha_n)\beta_n}{1 + (1 - \rho)\alpha_n} \right] \|Sz - z\| \|x_{n+1} - z\| \\ &= \left[1 - \frac{2(1 - \rho)\alpha_n}{1 + (1 - \rho)\alpha_n} \right] \|x_n - z\|^2 + \left[\frac{2(1 - \rho)\alpha_n}{1 + (1 - \rho)\alpha_n} \right] \\ &\quad \times \left\{ \frac{1}{1 - \rho} \langle f(z) - z, x_{n+1} - z \rangle + \frac{(1 - \alpha_n)\beta_n}{(1 - \rho)\alpha_n} \right. \\ &\quad \left. \times \|Sz - z\| \|x_{n+1} - z\| \right\}. \end{aligned}$$

Let $\gamma_n = \frac{2(1 - \rho)\alpha_n}{1 + (1 - \rho)\alpha_n}$ and $\delta_n = \frac{2(1 - \rho)\alpha_n}{1 + (1 - \rho)\alpha_n} \left\{ \frac{1}{1 - \rho} \langle f(z) - z, x_{n+1} - z \rangle + \frac{(1 - \alpha_n)\beta_n}{(1 - \rho)\alpha_n} \cdot \|Sz - z\| \|x_{n+1} - z\| \right\}$ for all $n \geq 1$. Since

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \left\{ \frac{1}{1 - \rho} \langle f(z) - z, x_{n+1} - z \rangle + \frac{(1 - \alpha_n)\beta_n}{(1 - \rho)\alpha_n} \right. \\ &\quad \left. \times \|Sz - z\| \|x_{n+1} - z\| \right\} \leq 0, \\ &\sum_{i=1}^{\infty} \alpha_n = \infty \text{ and } \frac{2(1 - \rho)\alpha_n}{1 + (1 - \rho)\alpha_n} \geq (1 - \rho)\alpha_n, \text{ we have} \end{aligned}$$

$$\sum_{n=1}^{\infty} \gamma_n = \infty \text{ and } \limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0.$$

Hence, by Lemma 5, we conclude that $x_n \rightarrow z$ as $n \rightarrow \infty$. \square

Remark 1. In the iterative scheme (30), if we set $f \equiv 0$, then we get $x_n \rightarrow z = P_{\mathcal{F}}0$. In this case, from (31), it follows that

$$\langle z, z - x \rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

That is

$$\|z\|^2 \leq \langle z, x \rangle \leq \|z\|\|x\|, \quad \forall x \in \mathcal{F}.$$

Therefore, the point z is the unique solution to the following quadratic minimization problem:

$$z = \arg \min_{x \in \mathcal{F}} \|x\|^2.$$

By changing the restrictions on parameters in Theorem 1, we obtain the following results.

Theorem 2. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $f : C \rightarrow H$ be a ρ -contraction mapping, $S : C \rightarrow C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \rightarrow C$ be a countable family of k_i -strict pseudo-contraction mappings and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, and $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C[\beta_n Sx_n + (1 - \beta_n)x_n] = \beta_n Sx_n + (1 - \beta_n)x_n \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) V_i y_n], \quad \forall n \geq 1 \end{cases} \quad (34)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $V_i = k_i I + (1 - k_i)T_i$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0, \infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \mathcal{F}. \quad (35)$$

Proof. First, we shows that (49) has the unique solution. Let x' and x^* be two solutions. Then, since x' is solution, for $y = x^*$ one has

$$\langle (I - f)x', x' - x^* \rangle \leq \tau \langle (I - S)x', x^* - x' \rangle \quad (36)$$

and

$$\langle (I - f)x^*, x^* - x' \rangle \leq \tau \langle (I - S)x^*, x' - x^* \rangle. \quad (37)$$

Adding (36) and (37), we obtain

$$\begin{aligned} (1 - \rho)\|x' - x^*\|^2 &\leq \langle (I - f)x' - (I - f)x^*, x' - x^* \rangle \\ &\leq -\rho \langle (I - S)x' - (I - S)x^*, x' - x^* \rangle \leq 0 \end{aligned}$$

so $x' = x^*$. Also now the condition (H2) with $0 < \tau < \infty$ implies (H1) so the sequence $\{x_n\}$ is bounded. Moreover, since (H8) implies (H6) and (H7), then $\{x_n\}$ is asymptotically regular. Similarly, by Proposition 2, the weak cluster points set of x_n , i.e., $\omega_w(x_n)$, is a subset of \mathcal{F} .

From (20)-(24), we observe that

$$\begin{aligned} \frac{\|x_{n+1} - x_n\|}{\beta_n} &\leq \frac{\|u_n - u_{n-1}\|}{\beta_n} \leq [1 - (1 - \rho)\alpha_n] \frac{\|x_n - x_{n-1}\|}{\beta_n} \\ &\quad + M \left[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} \right] \\ &= [1 - (1 - \rho)\alpha_n] \frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} \\ &\quad + [1 - (1 - \rho)\alpha_n] \|x_n - x_{n-1}\| \left[\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right] \\ &\quad + M \left[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} \right] \\ &\leq [1 - (1 - \rho)\alpha_n] \frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} \\ &\quad + \|x_n - x_{n-1}\| \left[\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right] \\ &\quad + M \left[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} \right] \\ &\leq [1 - (1 - \rho)\alpha_n] \frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \alpha_n K \|x_n - x_{n-1}\| \\ &\quad + M \left[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} \right] \\ &\leq [1 - (1 - \rho)\alpha_n] \frac{\|u_n - u_{n-1}\|}{\beta_{n-1}} + \alpha_n K \|x_n - x_{n-1}\| \\ &\quad + M \left[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} \right]. \end{aligned}$$

Let $\gamma_n = (1 - \rho)\alpha_n$ and $\delta_n = \alpha_n K \|x_n - x_{n-1}\| + M \left[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} \right]$. From condition (H3) and (H8), we have

$$\sum_{i=1}^{\infty} \gamma_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} = 0.$$

By Lemma 5, we obtain

$$\lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x_n\|}{\beta_n} = 0,$$

$$\lim_{n \rightarrow \infty} \frac{\|u_{n+1} - u_n\|}{\beta_n} = \lim_{n \rightarrow \infty} \frac{\|u_{n+1} - u_n\|}{\alpha_n} = 0.$$

From (34), we have

$$\begin{aligned} x_n - x_{n-1} &= (1 - \alpha_n)x_n - \left[P_C[u_n] - u_n + \alpha_n f(x_n) \right. \\ &\quad \left. + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)(V_i y_n - y_n) + (1 - \alpha_n)y_n \right] \\ &= (1 - \alpha_n)\beta_n(x_n - Sx_n) + (u_n - P_C[u_n]) \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)(y_n - V_i y_n) + \alpha_n(x_n - f(x_n)). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{x_n - x_{n-1}}{(1 - \alpha_n)\beta_n} &= (x_n - Sx_n) + \frac{1}{(1 - \alpha_n)\beta_n}(u_n - P_C[u_n]) \\ &\quad + \frac{1}{(1 - \alpha_n)\beta_n} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)(y_n - V_i y_n) \\ &\quad + \frac{\alpha_n}{(1 - \alpha_n)\beta_n}(x_n - f(x_n)). \end{aligned}$$

Let $v_n = \frac{x_n - x_{n-1}}{(1 - \alpha_n)\beta_n}$. For all $z \in \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{i=1}^{\infty} F(V_i)$, we get

$$\begin{aligned} \langle v_n, x_n - z \rangle &= \frac{1}{(1 - \alpha_n)\beta_n} \langle u_n - P_C[u_n], P_C[u_{n-1}] - z \rangle \\ &\quad + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \langle (I - f)x_n, x_n - z \rangle \\ &\quad + \langle x_n - Sx_n, x_n - z \rangle + \frac{1}{(1 - \alpha_n)\beta_n} \\ &\quad \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle y_n - V_i y_n, x_n - z \rangle. \end{aligned} \quad (38)$$

By Lemma 4, we have

$$\begin{aligned} \langle x_n - Sx_n, x_n - z \rangle &= \langle (I - S)x_n - (I - S)z, x_n - z \rangle \\ &\quad + \langle (I - S)z, x_n - z \rangle \geq \langle (I - S)z, x_n - z \rangle, \end{aligned} \quad (39)$$

$$\begin{aligned} \langle (I - f)x_n, x_n - z \rangle &= \langle (I - f)x_n - (I - f)z, x_n - z \rangle \\ &\quad + \langle (I - f)z, x_n - z \rangle \geq (1 - \rho)\|x_n - z\|^2 \\ &\quad + \langle (I - f)z, x_n - z \rangle \end{aligned} \quad (40)$$

and

$$\begin{aligned} \langle y_n - V_i y_n, x_n - z \rangle &= \langle (I - V_i)y_n - (I - V_i)z, x_n - y_n \rangle \\ &\quad + \langle (I - V_i)y_n - (I - V_i)z, y_n - z \rangle \\ &\geq \langle (I - V_i)y_n - (I - V_i)z, x_n - y_n \rangle \\ &= \beta_n \langle (I - V_i)y_n, x_n - Sx_n \rangle, \quad \forall i \geq 1. \end{aligned} \quad (41)$$

By Lemma 3(1), we obtain

$$\begin{aligned} \langle u_n - P_C[u_n], P_C[u_{n-1}] - z \rangle &= \langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n] \rangle \\ &\quad + \langle u_n - P_C[u_n], P_C[u_n] - z \rangle \\ &\geq \langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n] \rangle. \end{aligned} \quad (42)$$

Now, from (38)-(42), it follows that

$$\begin{aligned} \|x_n - z\|^2 &\leq \frac{(1 - \alpha_n)\beta_n}{(1 - \rho)\alpha_n} [\langle v_n, x_n - z \rangle - \langle (I - S)z, x_n - z \rangle] \\ &\quad + \frac{\|u_{n-1} - u_n\|}{(1 - \rho)\alpha_n} \|u_n - P_C[u_n]\| \\ &\quad - \frac{1}{1 - \rho} \langle (I - f)z, x_n - z \rangle - \frac{\beta_n}{(1 - \rho)\alpha_n} \\ &\quad \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle (I - V_i)y_n, x_n - Sx_n \rangle, \end{aligned} \quad (43)$$

since $v_n \rightarrow 0$ and $(I - V_i)y_n \rightarrow 0$, as $n \rightarrow \infty$, then every weak cluster point of $\{x_n\}$ is also a strong cluster point. By Proposition 2, $\{x_n\}$ is bounded, thus there exists

a subsequence $\{x_{n_k}\}$ converging to x^* . For all $z \in \mathcal{F}$ by (38), we compute

$$\begin{aligned} \langle (I-f)x_{n_k}, x_{n_k} - z \rangle &= \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}} \langle v_{n_k}, x_{n_k} - z \rangle \\ &\quad - \frac{1}{\alpha_{n_k}} \langle u_{n_k} - P_C[u_{n_k}], P_C[u_{n_k-1}] - z \rangle \\ &\quad - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}} \langle x_{n_k} - Sx_{n_k}, x_{n_k} - z \rangle \\ &\quad - \frac{1}{(\alpha_{n_k})} \sum_{i=1}^{n_k} (\alpha_{i-1} - \alpha_i) \\ &\quad \times \langle y_{n_k} - V_i y_{n_k}, x_{n_k} - z \rangle \\ &\leq \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}} \langle v_{n_k}, x_{n_k} - z \rangle - \frac{\beta_{n_k}}{(\alpha_{n_k})} \\ &\quad \sum_{i=1}^{n_k} (\alpha_{i-1} - \alpha_i) \langle (I - V_i)y_{n_k}, x_{n_k} - Sx_{n_k} \rangle \\ &\quad - \frac{1}{\alpha_{n_k}} \|u_{n_k-1} - u_{n_k}\| \|u_{n_k} - P_C[u_{n_k}]\| \\ &\quad - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}} \langle (I - S)z, x_{n_k} - z \rangle. \end{aligned} \tag{44}$$

Since $v_n \rightarrow 0, (I - V_i)y_n \rightarrow 0$ for all $i \geq 1$, and $\|u_{n-1}\|/\alpha_n \rightarrow 0$, letting $k \rightarrow \infty$ in (44), we obtain

$$\langle (I-f)x^*, x^* - z \rangle \leq -\tau \langle (I - S)z, x^* - z \rangle, \quad \forall z \in \mathcal{F}.$$

Since (49) has the unique solution, it follows that $\omega_w(x_n) = \{x^*\}$. Since every weak cluster point of $\{x_n\}$ is also a strong cluster point, we conclude that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. This completes the proof.

If we take $T_i = T$, for all $i \geq 1$, where $T : C \rightarrow C$ is a k -strict pseudo-contraction mapping in Theorem 1, then we get the following result: \square

Corollary 2. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $f : C \rightarrow H$ be a ρ -contraction mapping, $S : C \rightarrow H$ be a nonexpansive mapping and $T : C \rightarrow C$ be a k -strict pseudo-contraction mapping such that $F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C[\beta_n Sx_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1 - \alpha_n)Vy_n], \quad \forall n \geq 1, \end{cases} \tag{45}$$

where $V = kI + (1 - k)T, \{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ are sequences satisfying the conditions (H2) with $\tau = 0$,

(H3), either (H4) and (H5), or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in F(T)$, which is the unique solution of the variational inequality:

$$\langle (I-f)z, x - z \rangle \geq 0, \quad \forall x \in F(T).$$

Taking $k_i = 0$, for all $i \geq 1$ in Theorem 1, then we get the following result:

Corollary 3. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $f : C \rightarrow H$ be a ρ -contraction mapping, $S : C \rightarrow H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^\infty : C \rightarrow C$ be a countable family of nonexpansive mappings and $\mathcal{F} = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$. Let $\alpha_0 = 1, x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C[\beta_n Sx_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \tag{46}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5), or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle (I-f)z, x - z \rangle \geq 0, \quad \forall x \in \mathcal{F}.$$

If we take $k = 0$ in Corollary 2, then we get the following result:

Corollary 4. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $f : C \rightarrow H$ be a ρ -contraction mapping, $S : C \rightarrow H$ be a nonexpansive mapping and $T : C \rightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C[\beta_n Sx_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1 - \alpha_n)Ty_n], \quad \forall n \geq 1, \end{cases} \tag{47}$$

where $\{\alpha_n\} \subset (0, 1), \{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5), or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in F(T)$, which is the unique solution of the variational inequality:

$$\langle (I-f)z, x - z \rangle \geq 0, \quad \forall x \in F(T).$$

If we take $T_i = T$, for all $i \geq 1$, where $T : C \rightarrow C$ is a k -strict pseudo-contraction mapping in Theorem 2, then we obtain the following result:

Corollary 5. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $f : C \rightarrow H$ be a ρ -contraction mapping, $S : C \rightarrow C$ be a nonexpansive mapping and $T : C \rightarrow C$ be a k -strict pseudo-contraction mapping and $\mathcal{F} = F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1 - \alpha_n)Vy_n], \quad \forall n \geq 1, \end{cases} \quad (48)$$

where $V = kI + (1 - k)T$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0, \infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \mathcal{F}. \quad (49)$$

If we take $k_i = 0$, for all $i \geq 1$ in Theorem 2, then we get the following result:

Corollary 6. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $f : C \rightarrow H$ be a ρ -contraction mapping, $S : C \rightarrow C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^\infty : C \rightarrow C$ be a countable family of nonexpansive mappings and $\mathcal{F} = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \quad (50)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0, \infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \mathcal{F}. \quad (51)$$

If $k = 0$ in Corollary 5, then we get the following Corollary:

Corollary 7. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $f : C \rightarrow H$ be a ρ -contraction mapping, $S, T : C \rightarrow C$ be nonexpansive

mappings and $\mathcal{F} = F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1 - \alpha_n)Ty_n], \quad \forall n \geq 1, \end{cases} \quad (52)$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0, \infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \mathcal{F}. \quad (53)$$

Remark 2. Prototypes for the iterative parameters are, for example, $\alpha_n = n^{-\theta}$ and $\beta_n = n^{-\omega}$ (with $\theta, \omega > 0$). Since $|\alpha_n - \alpha_{n-1}| \approx n^{-\theta}$ and $|\beta_n - \beta_{n-1}| \approx n^{-\omega}$, it is not difficult to prove that (H8) is satisfied for $0 < \theta, \omega < 1$ and (H9) is satisfied if $\theta + \omega \leq 1$.

Remark 3. Theorem 1 and Theorem 2 extend and improve the result of Gu et al. (2011) from the countable family of nonexpansive mappings to more general the countable family of strictly pseudo contraction mappings.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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