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The modified alternative (G′/G)-expansion method to nonlinear evolution equation: application to the (1+1)-dimensional Drinfel'd-Sokolov-Wilson equation

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Abstract

Over the years, (G′/G)–expansion method is employed to generate traveling wave solutions to various wave equations in mathematical physics. In the present paper, the alternative (G′/G)–expansion method has been further modified by introducing the generalized Riccati equation to construct new exact solutions. In order to illustrate the novelty and advantages of this approach, the (1+1)-dimensional Drinfel'd-Sokolov-Wilson (DSW) equation is considered and abundant new exact traveling wave solutions are obtained in a uniform way. These solutions may be imperative and significant for the explanation of some practical physical phenomena. It is shown that the modified alternative (G′/G)–expansion method an efficient and advance mathematical tool for solving nonlinear partial differential equations in mathematical physics.

Keywords: (G′/G)-expansion method; Travelling wave solutions; DSW equation; Nonlinear evolution equations

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Introduction

After the observation of soliton phenomena by John Scott Russell in 1834 (Wazwaz 2009) and since the KdV equation was solved by Gardner et al. (1967) by inverse scattering method, finding exact solutions of nonlinear evolution equations (NLEEs) has turned out to be one of the most exciting and particularly active areas of research. The appearance of solitary wave solutions in nature is quite common. Bell-shaped sech-solutions and kink-shaped tanh-solutions model wave phenomena in elastic media, plasmas, solid state physics, condensed matter physics, electrical circuits, optical fibers, chemical kinematics, fluids, bio-genetics etc. The traveling wave solutions of the KdV equation and the Boussinesq equation which describe water waves are well-known examples. Apart from their physical relevance, the closed-form solutions of NLEEs if available facilitate the numerical solvers in comparison, and aids in the stability analysis. In soliton theory, there are several techniques to deal with the problems of solitary wave solutions for NLEEs, such as, Hirota's bilinear transformation (Hirota 1971), Backlund transformation (Rogers & Shadwick 1982), improved homotopy perturbation

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(Jafari & Aminataei 2010), Darboux transformation (Zhaqilao 2010), tanh-function (Malfliet 1992), homogeneous balance (Wang 1996), Jacobi elliptic function (Liu et al. 2001; Ali 2011), F-expansion (Zhou et al. 2003) and Exp-function (He & Wu 2006; Abdou et al. 2007; Akbar & Ali 2011; Naher et al. 2012). It is to be highlighted that Marinca and Herişanu (2011) applied a new approach for calculating a kind of explicit exact solution of nonlinear differential equations and in the simiar context obtained exact solutions of the Duffing and double-well Duffing equations. They implemented the new proposed procedure by using a quotient trigonometric function expansion method and also proved that the introduced method could be easily applied to solve other nonlinear differential equations.

Recently, Wang et al. (2008) established a widely used direct and concise method called the (G'/G)-expansion method for obtaining the exact travelling wave solutions of NLEEs, where $G(\xi)$ satisfies the second order linear ordinary differential equation (ODE) $G'' + \lambda\, G' + \mu G = 0$, where λ and μ are arbitrary constants. Applications of the (G'/G)-expansion method can be found in the articles (Bekir 2008; Naher et al. 2011; Akbar et al. 2012; Kol & Tabi 2011; Zayed & Gepreel 2009; Zayed 2009a; Zhang et al. 2008a; Zhang et al. 2008b; Abazari 2010; Liu et al. 2010) for better understanding.

In order to establish the effectiveness and reliability of the (G'/G)-expansion method and to expand the possibility of its application, further research has been carried out by several researchers. For instance, Zhang et al. (2010) presented an improved (G'/G)-expansion method to seek more general traveling wave solutions. Zayed (2009b) presented a new approach of the (G'/G)-expansion method where $G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2 = e_2G^4(\xi) + e_1G^2(\xi) + e_0$, e_2, e_1, e_0 are arbitrary constants, and obtained new exact solutions. Zayed (2011) again presented an alternative approach of this method in which $G(\xi)$ satisfies the Riccati equation $G'(\xi) = A + B\, G^2(\xi)$, where A and B are arbitrary constants.

Still, substantial work has to be done in order for the (G'/G)-expansion method to be well established, since every nonlinear equation has its own physically significant rich structure. For finding the new exact solutions of NLEEs, it is important to present various method and ansatz, but it seems to be more important how to obtain more new exact solutions to NLEEs under the known method and ansatz. In the present article, we further modify the alternative (G'/G)-expansion method (presented by Zayed (2011)) by introducing the generalized Riccati equation mapping, its twenty seven solutions and constructed abundant new traveling wave solutions of the DSW equation.

The method

Suppose the general nonlinear partial differential equation,

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0 \tag{1}$$

where $u=u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved. The main steps of the modified alternative (G'/G)-expansion method combined with the generalized Riccati equation mapping are as follows:

Step 1: The travelling wave variable ansatz

$$u(x,t) = u(\xi), \quad \xi = x - Vt \tag{2}$$

where V is the speed of the traveling wave, permits us to transform the Equation (1) into an ODE:

$$Q\left(u, u', u'', \cdots\right) = 0 \tag{3}$$

where the superscripts stands for the ordinary derivatives with respect to ξ.

Step 2: Suppose the traveling wave solution of Equation (3) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = \sum_{n=0}^{m} a_n \left(\frac{G'}{G}\right)^n, \quad a_m \neq 0 \tag{4}$$

where $G'/G(\xi)$ satisfies the generalized Riccati equation,

$$G' = r + pG + qG^2, \tag{5}$$

where a_n $(n = 0, 1, 2, \cdots, m)$, r, p and q are arbitrary constants to be determined later.

The generalized Riccati Equation (5) has twenty seven solutions (Zhu, 2008) as follows:

Family 1: When $p^2 - 4\,q\,r < 0$ and $pq \neq 0$ (or $r\,q \neq 0$), the solutions of Equation (5) are,

$$G_1 = \frac{1}{2q}\left[-p + \sqrt{4qr-p^2}\tan\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)\right],$$

$$G_2 = -\frac{1}{2q}\left[p + \sqrt{4qr-p^2}\cot\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)\right],$$

$$G_3 = \frac{1}{2q}\left[-p + \sqrt{4qr-p^2}\left(\tan\left(\sqrt{4qr-p^2}\xi\right) \pm \sec\left(\sqrt{4qr-p^2}\xi\right)\right)\right],$$

$$G_4 = -\frac{1}{2q}\left[p + \sqrt{4qr-p^2}\left(\cot\left(\sqrt{4qr-p^2}\xi\right) \pm \csc\left(\sqrt{4qr-p^2}\xi\right)\right)\right],$$

$$G_5 = \frac{1}{4q}\left[-2p + \sqrt{4qr-p^2}\left(\tan\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right) - \cot\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)\right)\right],$$

$$G_6 = \frac{1}{2q}\left[-p + \frac{\sqrt{\left(A^2-B^2\right)\left(4qr-p^2\right)} - A\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\xi\right)}{A\sin\left(\sqrt{4qr-p^2}\xi\right) + B}\right],$$

$$G_7 = \frac{1}{2q}\left[-p + \frac{\sqrt{\left(A^2-B^2\right)\left(4qr-p^2\right)} + A\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\xi\right)}{A\sin\left(\sqrt{4qr-p^2}\xi\right) + B}\right],$$

where A and B are two non-zero real constants and satisfies the condition $A^2 - B^2 > 0$.

$$G_8 = \frac{-2r\cos\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)}{\sqrt{4qr-p^2}\sin\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)+p\cos\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)},$$

$$G_9 = \frac{2r\sin\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)}{-p\sin\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)+\sqrt{(4qr-p^2)}\cos\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)},$$

$$G_{10} = \frac{-2r\cos\left(\sqrt{4qr-p^2}\xi\right)}{\sqrt{(4qr-p^2)}\sin\left(\sqrt{4qr-p^2}\xi\right)+p\cos\left(\sqrt{4qr-p^2}\xi\right)\pm\sqrt{(4qr-p^2)}},$$

$$G_{11} = \frac{2r\sin\left(\sqrt{4qr-p^2}\xi\right)}{-p\sin\left(\sqrt{4qr-p^2}\xi\right)+\sqrt{(4qr-p^2)}\cos\left(\sqrt{4qr-p^2}\xi\right)\pm\sqrt{(4qr-p^2)}},$$

$$G_{12} = \frac{4r\sin\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)}{-2p\sin\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)+2\sqrt{(4qr-p^2)}\cos^2\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)-\sqrt{(4qr-p^2)}}.$$

Family 2: When $p^2 - 4\,q\,r > 0$ and $pq \neq 0$ (or $r\,q \neq 0$), the solutions of Equation (5) are,

$$G_{13} = -\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\tanh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)\right],$$

$$G_{14} = -\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\coth\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)\right],$$

$$G_{15} = -\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\left(\tanh\left(\sqrt{p^2-4qr}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{16} = -\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\left(\coth\left(\sqrt{p^2-4qr}\xi\right)\pm\,\mathrm{csch}\left(\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{17} = -\frac{1}{4q}\left[2p+\sqrt{p^2-4qr}\left(\tanh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)+\coth\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{18} = \frac{1}{2q}\left[-p+\frac{\sqrt{\left(A^2+B^2\right)(p^2-4qr)}-A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)}{A\sinh\left(\sqrt{p^2-4qr}\xi\right)+B}\right],$$

$$G_{19} = \frac{1}{2q}\left[-p-\frac{\sqrt{\left(B^2-A^2\right)(p^2-4qr)}+A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)}{A\sinh\left(\sqrt{p^2-4qr}\xi\right)+B}\right],$$

where A and B are two non-zero real constants and satisfies the condition $B^2 - A^2 > 0$.

$$G_{20} = \frac{2r\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)-p\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)},$$

$$G_{21} = \frac{2r\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)-p\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)},$$

$$G_{22} = \frac{2r\cosh\left(\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}\xi\right)-p\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm i\sqrt{p^2-4qr}},$$

$$G_{23} = \frac{2r\sinh\left(\sqrt{p^2-4qr}\xi\right)}{-p\sinh\left(\sqrt{p^2-4qr}\xi\right)+\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm\sqrt{p^2-4qr}},$$

$$G_{24} = \frac{4r\sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)}{-2p\sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)+2\sqrt{p^2-4qr}\cosh^2\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)-\sqrt{p^2-4qr}}.$$

Family 3: When r=0 and $pq\neq 0$, the solutions of Equation (5) are,

$$G_{25} = \frac{-pd}{q[d+\cosh(p\xi)-\sinh(p\xi)]},$$

$$G_{26} = -\frac{p[\cosh(p\xi)+\sinh(p\xi)]}{q[d+\cosh(p\xi)+\sinh(p\xi)]},$$

where d is an arbitrary constant.

Family 4: When $q\neq 0$ and r=p=0, the solution of Equation (5) is,

$$G_{27} = -\frac{1}{q\xi + c_1},$$

where c_1 is an arbitrary constant.

Step 3: To determine the positive integer m, substitute Equation (4) along with Equation (5) into Equation (3) and then consider homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Equation (3).

Step 4: Substituting Equation (4) along with Equation (5) into Equation (3) together with the value of m obtained in step 3, we obtain polynomials in G^i and G^{-i} ($i = 0, 1, 2, 3\cdots$) and vanishing each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for a_n p, q, r and V.

Step 5: Suppose the value of the constants a_n p, q, r and V can be determined by solving the set of algebraic equations obtained in step 4. Since the general solutions of

Equation (5) are known, substituting, a_n p, q, r and V into Equation (4), we obtain new exact traveling wave solutions of the nonlinear evolution Equation (1).

Some new traveling wave solutions of the DSW equation

In this section, the modified alternative (G'/G)-expansion method is employed to construct some new traveling wave solutions of the (1+1)-dimensional Drinfel'd-Sokolov-Wilson (DSW) equation which is very important nonlinear evolution equation in mathematical physics and engineering and have been paid attention by many researchers. Some exact solutions of the DSW equation were found in the literature. In general, the solutions of the DSW equation have been obtained by means of an ansatz method. Included in the methods are the elliptic-function (Chen & Zhang 2003; Liu et al. 2005), Exp-function (He et al. 2010), Darboux transformation (Guo & Wu 2010), improved F-expansion (Zha & Zhi 2008), Variational iteration (Zhang 2011) and Adomian's decomposition (Inc 2006). It is to be highlighted that Marinca et. al. (2011) presented quotient trigonometric function expansion method to find explicit and exact solutions to cubic Duffing and double-well Duffing equations. Moreover, a detailed study is made by Yang (2012) on local fractional differential equations and its Applications, Local Fractional Functional Analysis and its Applications along with local fractional variation iteration and local fractional Fourier series methods. He (2012) has also given a comprehensive analysis of Asymptotic methods for solitary solutions and compactons. Inspired and motivated by the ongoing research in this area, we apply the modified alternative (G'/G)-expansion method for searching its new solitary wave solutions. Let us consider the DSW equation:

$$2v_{xxx} + 2uv_x + u_x v - v_t = 0 \tag{6}$$

$$3vv_x - u_t = 0. \tag{7}$$

Now, we use the wave transformation Equation (2) into Equations (6) and (7), which yield:

$$2v''' + 2uv' + vu' + Vv' = 0, \tag{8}$$

$$3vv' + Vu' = 0. \tag{9}$$

According to step 3, the solution of Equations (8) and (9) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2 + \cdots + a_m\left(\frac{G'}{G}\right)^m, \quad a_m \neq 0 \tag{10}$$

and

$$v(\xi) = b_0 + b_1\left(\frac{G'}{G}\right) + b_2\left(\frac{G'}{G}\right)^2 + \cdots + b_n\left(\frac{G'}{G}\right)^n \quad b_n \neq 0 \tag{11}$$

where a_i, ($i = 0, 1, 2, \cdots, m$) and b_j, ($j = 0, 1, 2, \cdots, n$) all are constants to be determined and $G'/G(\xi)$ satisfies the generalized Riccati Equation (5). Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Equations (8) and (9), we obtain m=2 and n=1.

Therefore, solution Equations (10) and (11) take the form respectively

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \ a_2 \neq 0 \tag{12}$$

$$v(\xi) = b_0 + b_1\left(\frac{G'}{G}\right), b_1 \neq 0 \tag{13}$$

By means of Equation (5), Equations (12) and (13) can be rewritten respectively as,

$$u(\xi) = a_0 + a_1\left(p + rG^{-1} + qG\right) + a_2\left(p + rG^{-1} + qG\right)^2 \tag{14}$$

and

$$v(\xi) = b_0 + b_1\left(p + rG^{-1} + qG\right). \tag{15}$$

Substituting Equations (14) and (15) into Equations (8) and (9), the left hand sides of these equations are converted into polynomials in G^i and G^{-i}, $(i = 0, 1, 2, 3, \cdots)$. Setting each coefficient of these polynomials to zero, we obtain a set of simultaneous algebraic equations for a_0, a_1, a_2, b_0, b_1, p, q, r and V as follows:

$$\begin{aligned}
&24b_1p + 3a_1b_1 + 2a_2b_0 + 12a_2b_1p = 0, \ \ 2a_2V + 3b_1^2 = 0, \ \ a_2b_1 + 3b_1 = 0,\\
&a_1b_0 + 6a_1b_1p + Vb_1 + 4a_2b_0p + 16b_1qr + 2a_0b_1 + 8a_2b_1qr + 12a_2b_1p^2 + 14b_1p^2 = 0,\\
&3a_1b_1rq + 4a_2b_1p^3 + Vb_1p + 3a_1b_1p^2 + 2a_2b_0p^2 + 2a_0b_1p + 2b_1p^3 + a_1b_0p\\
&+2a_2b_0rq + 16b_1pqr + 12a_2pqr = 0,\\
&Vb_1p + a_1b_0p + 2a_0b_1p + 12a_2b_1pqr + 4a_2b_1p^3 + 3a_1b_1p^2 + 2b_1p^3 + 2a_2b_0p^2\\
&+2a_2b_0rq + 3a_1b_1qr + 16b_1pqr = 0,\\
&a_1b_0 + 8a_2b_1rq + 6a_1b_1p + 14b_1p^2 + 4a_2b_0p + 16b_1qr + 12a_2b_1p^2 + 2a_0b_1 + Vb_1 = 0,\\
&3a_1b_1 + 24b_1p + 2a_2b_0 + 12a_2b_1 \ p = 0, \ \ Va_1 + 4a_2Vp + 3b_0b_1 + 6b_1^2 \ p = 0,\\
&3b_0b_1p + 3b_1^2rq + 3b_1^2p^2 + a_1pV + 2a_2p^2V + 2a_2rqV\\
&= 0, \ \ 3b_0b_1 + 4a_2pV + 6b_1^2p + a_1V = 0,\\
&3b_0b_1p + 3b_1^2p^2 + a_1pV + 2a_2p^2V + 3b_1^2qr + 2a_2qrV = 0.
\end{aligned} \tag{16}$$

Solving the over-determined set of algebraic equations by using the symbolic computation software, such as, Maple, we obtain

$$a_2 = -3, \ \ a_1 = 3p, \ \ a_0 = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr, \ \ b_1 = b_1, \ \ b_0 = -\frac{1}{2}b_1p, \ \ V = \frac{1}{2}b_1^2 \tag{17}$$

where b_1, p, q and r are arbitrary constants.

Now on the basis of the solutions of Equation (5), we obtain some new types of solutions of Equations (6) and (7).

Family 1

When $p^2 - 4\ q\ r < 0$ and $pq \neq 0$ (or $r\ q \neq 0$), the periodic form solutions of Equations (6) and (7) are:

$$u_1 = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr + 3p\left(\frac{2\Delta^2\ \sec^2(\Delta\xi)}{-p + 2\Delta\tan(\Delta\xi)}\right) - 3\left(\frac{2\Delta^2\ \sec^2(\Delta\xi)}{-p + 2\Delta\tan(\Delta\xi)}\right)^2,$$

$$v_1 = -\frac{1}{2}b_1 p + b_1\left(\frac{2\Delta^2\ \sec^2(\Delta\xi)}{-p + 2\Delta\tan(\Delta\xi)}\right),$$

where $\Delta = \frac{1}{2}\sqrt{4qr - p^2}$, $\xi = x - \frac{1}{2}b_1^2 t$ and b_1, p, q, r are arbitrary constants.

$$u_2 = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr - 3p\left(\frac{2\Delta^2\ \csc^2(\Delta\xi)}{p + 2\Delta\cot(\Delta\xi)}\right) - 3\left(\frac{2\Delta^2\ \csc^2(\Delta\xi)}{p + 2\Delta\cot(\Delta\xi)}\right)^2,$$

$$v_2 = -\frac{1}{2}b_1 p - b_1\left(\frac{2\Delta^2\ \csc^2(\Delta\xi)}{p + 2\Delta\cot(\Delta\xi)}\right),$$

$$u_3 = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr + 3p\left(\frac{4\Delta^2\sec(2\Delta\xi)\,(1 \pm \sin(2\Delta\xi))}{-p\cos(2\Delta\xi) + 2\Delta\sin(2\Delta\xi) \pm 2\Delta}\right) - 3\left(\frac{4\Delta^2\sec(2\Delta\xi)\,(1 \pm \sin(2\Delta\xi))}{-p\cos(2\Delta\xi) + 2\Delta\sin(2\Delta\xi) \pm 2\Delta}\right)^2,$$

$$v_3 = -\frac{1}{2}b_1 p + b_1\left(\frac{4\Delta^2\sec(2\Delta\xi)\,(1 \pm \sin(2\Delta\xi))}{-p\cos(2\Delta\xi) + 2\Delta\sin(2\Delta\xi) \pm 2\Delta}\right),$$

$$u_4 = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr - 3p\left(\frac{4\Delta^2\csc(2\Delta\xi)\,(1 \pm \cos(2\Delta\xi))}{p\sin(2\Delta\xi) + 2\Delta\cos(2\Delta\xi) \pm 2\Delta}\right) - 3\left(\frac{4\Delta^2\csc(2\Delta\xi)\,(1 \pm \cos(2\Delta\xi))}{p\sin(2\Delta\xi) + 2\Delta\cos(2\Delta\xi) \pm 2\Delta}\right)^2,$$

$$v_4 = -\frac{1}{2}b_1 p - b_1\left(\frac{4\Delta^2\csc(2\Delta\xi)\,(1 \pm \cos(2\Delta\xi))}{p\sin(2\Delta\xi) + 2\Delta\cos(2\Delta\xi) \pm 2\Delta}\right),$$

$$u_5 = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr - 3p\left(\frac{2\Delta^2\csc(\Delta\xi)}{p\sin(\Delta\xi) + 2\Delta\cos(\Delta\xi)}\right) - 3\left(\frac{2\Delta^2\csc(\Delta\xi)}{p\sin(\Delta\xi) + 2\Delta\cos(\Delta\xi)}\right)^2,$$

$$v_5 = -\frac{1}{2}b_1 p - b_1\left(\frac{2\Delta^2\csc(\Delta\xi)}{p\sin(\Delta\xi) + 2\Delta\cos(\Delta\xi)}\right),$$

$$u_6 = -3p\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)-B\sin(2\Delta\xi)-A\right\}\{A\sin(2\Delta\xi)+B\}}{\{A^2\cos^2(2\Delta\xi)-A^2-B^2-2AB\sin(2\Delta\xi)\}\left\{pA\sin(2\Delta\xi)+2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right)$$
$$-3\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)-B\sin(2\Delta\xi)-A\right\}\{A\sin(2\Delta\xi)+B\}}{\{A^2\cos^2(2\Delta\xi)-A^2\,0-B^2-2AB\sin(2\Delta\xi)\}\left\{pA\sin(2\Delta\xi)+2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right)^2$$
$$-\frac{p^2}{4}-\frac{b_1^2}{4}+4qr,$$
$$v_6 = -\frac{1}{2}b_1p-b_1\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)-B\sin(2\Delta\xi)-A\right\}\{A\sin(2\Delta\xi)+B\}}{\{A^2\cos^2(2\Delta\xi)-A^2-B^2-2AB\sin(2\Delta\xi)\}\left\{pA\sin(2\Delta\xi)+2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right),$$

$$u_7 = -3p\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)+B\sin(2\Delta\xi)+A\right\}\{A\sin(2\Delta\xi)+B\}}{\{A^2\cos^2(2\Delta\xi)-A^2-B^2-2AB\sin(2\Delta\xi)\}\left\{pA\sin(2\Delta\xi)-2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right)$$
$$-3\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)+B\sin(2\Delta\xi)+A\right\}\{A\sin(2\Delta\xi)+B\}}{\{A^2\cos^2(2\Delta\xi)-A^2-B^2-2AB\sin(2\Delta\xi)\}\left\{pA\sin(2\Delta\xi)-2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right)^2$$
$$-\frac{p^2}{4}-\frac{b_1^2}{4}+4qr,$$
$$v_7 = -\frac{1}{2}b_1p-b_1\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)+B\sin(2\Delta\xi)+A\right\}\{A\sin(2\Delta\xi)+B\}}{\{A^2\cos^2(2\Delta\xi)-A^2-B^2-2AB\sin(2\Delta\xi)\}\left\{pA\sin(2\Delta\xi)-2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right),$$

where A and B are two non-zero real constants satisfies the condition $A^2 - B^2 > 0$.

$$u_8 = -\frac{p^2}{4}-\frac{b_1^2}{4}+4qr-3p\left(\frac{2\Delta^2\sec(\Delta\xi)\{p\cos(\Delta\xi)+2\Delta\sin(\Delta\xi)\}}{2(p^2-2rq)\cos^2(\Delta\xi)+4\Delta p\sin(\Delta\xi)\cos(\Delta\xi)+4\Delta^2}\right)$$
$$-3\left(\frac{2\Delta^2\sec(\Delta\xi)\{p\cos(\Delta\xi)+2\Delta\sin(\Delta\xi)\}}{2(p^2-2rq)\cos^2(\Delta\xi)+4\Delta p\sin(\Delta\xi)\cos(\Delta\xi)+4\Delta^2}\right)^2,$$

$$v_8 = -\frac{1}{2}b_1p-b_1\left(\frac{2\Delta^2\sec(\Delta\xi)\{p\cos(\Delta\xi)+2\Delta\sin(\Delta\xi)\}}{2(p^2-2rq)\cos^2(\Delta\xi)+4\Delta p\sin(\Delta\xi)\cos(\Delta\xi)+4\Delta^2}\right),$$

$$u_9 = -\frac{p^2}{4}-\frac{b_1^2}{4}+4qr+3p\left(\frac{2\Delta^2\csc(\Delta\xi)\{p\sin(\Delta\xi)-2\Delta\cos(\Delta\xi)\}}{2(p^2-2rq)\cos^2(\Delta\xi)+4\Delta p\sin(\Delta\xi)\cos(\Delta\xi)-p^2}\right)$$
$$-3\left(\frac{2\Delta^2\csc(\Delta\xi)\{p\sin(\Delta\xi)-2\Delta\cos(\Delta\xi)\}}{2(p^2-2rq)\cos^2(\Delta\xi)+4\Delta p\sin(\Delta\xi)\cos(\Delta\xi)-p^2}\right)^2,$$

$$v_9 = -\frac{1}{2}b_1p+b_1\left(\frac{2\Delta^2\csc(\Delta\xi)\{p\sin(\Delta\xi)-2\Delta\cos(\Delta\xi)\}}{2(p^2-2rq)\cos^2(\Delta\xi)+4\Delta p\sin(\Delta\xi)\cos(\Delta\xi)-p^2}\right),$$

$$u_{10} = -\frac{p^2}{4}-\frac{b_1^2}{4}+4qr-3p\left(\frac{2\Delta^2\sec(2\Delta\xi)\{1\pm\sin(2\Delta\xi)\}\{p\cos(2\Delta\xi)+2\Delta\sin(2\Delta\xi)\pm2\Delta\}}{(p^2-2rq)\cos^2(2\Delta\xi)+2\Delta\{1\pm\sin(2\Delta\xi)\}\{2\Delta\pm p\cos(2\Delta\xi)\}}\right)$$
$$-3\left(\frac{2\Delta^2\sec(2\Delta\xi)\{1\pm\sin(2\Delta\xi)\}\{p\cos(2\Delta\xi)+2\Delta\sin(2\Delta\xi)\pm2\Delta\}}{(p^2-2rq)\cos^2(2\Delta\xi)+2\Delta\{1\pm\sin(2\Delta\xi)\}\{2\Delta\pm p\cos(2\Delta\xi)\}}\right)^2,$$

$$v_{10} = -\frac{1}{2}b_1p-b_1\left(\frac{2\Delta^2\sec(2\Delta\xi)\{1\pm\sin(2\Delta\xi)\}\{p\cos(2\Delta\xi)+2\Delta\sin(2\Delta\xi)\pm2\Delta\}}{(p^2-2rq)\cos^2(2\Delta\xi)+2\Delta\{1\pm\sin(2\Delta\xi)\}\{2\Delta\pm p\cos(2\Delta\xi)\}}\right),$$

$$u_{11} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr \pm 3p\left(\frac{2\Delta^2 \csc(2\Delta\xi)\{-p\sin(2\Delta\xi) + 2\Delta\cos(2\Delta\xi) \pm 2\Delta\}}{(2rq-p^2)\cos(2\Delta\xi) - 2p\Delta\sin(2\Delta\xi) \pm 2qr}\right)$$
$$-3\left(\frac{2\Delta^2 \csc(2\Delta\xi)\{-p\sin(2\Delta\xi) + 2\Delta\cos(2\Delta\xi) \pm 2\Delta\}}{(2rq-p^2)\cos(2\Delta\xi) - 2p\Delta\sin(2\Delta\xi) \pm 2qr}\right)^2,$$

$$v_{11} = -\frac{1}{2}b_1 p \pm b_1\left(\frac{2\Delta^2 \csc(2\Delta\xi)\{-p\sin(2\Delta\xi) + 2\Delta\cos(2\Delta\xi) \pm 2\Delta\}}{(2rq-p^2)\cos(2\Delta\xi) - 2p\Delta\sin(2\Delta\xi) \pm 2qr}\right),$$

$$u_{12} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr + 3p\left(\frac{2\Delta^2 \csc(\Delta\xi)\{p\sin(\Delta\xi) - 2\Delta\cos(\Delta\xi)\}}{2(p^2-2rq)\cos^2(\Delta\xi) + 4\Delta p\sin(\Delta\xi)\cos(\Delta\xi) - p^2}\right)$$
$$-3\left(\frac{2\Delta^2 \csc(\Delta\xi)\{p\sin(\Delta\xi) - 2\Delta\cos(\Delta\xi)\}}{2(p^2-2rq)\cos^2(\Delta\xi) + 4\Delta p\sin(\Delta\xi)\cos(\Delta\xi) - p^2}\right)^2,$$

$$v_{12} = -\frac{1}{2}b_1 p + b_1\left(\frac{2\Delta^2 \csc(\Delta\xi)\{p\sin(\Delta\xi) - 2\Delta\cos(\Delta\xi)\}}{2(p^2-2rq)\cos^2(\Delta\xi) + 4\Delta p\sin(\Delta\xi)\cos(\Delta\xi) - p^2}\right).$$

Family 2

When $p^2 - 4\ q\ r > 0$ and $pq \neq 0$ (or $rq \neq 0$), the soliton and soliton-like solutions of Equations (6) and (7) are:

$$u_{13} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr + 3p\left(\frac{2\Omega^2 \operatorname{sech}^2(\Omega\xi)}{p + 2\Omega\tanh(\Omega\xi)}\right) - 3\left(\frac{2\Omega^2 \operatorname{sech}^2(\Omega\xi)}{p + 2\Omega\tanh(\Omega\xi)}\right)^2,$$

$$v_{13} = -\frac{1}{2}b_1 p + b_1\left(\frac{2\Omega^2 \operatorname{sech}^2(\Omega\xi)}{p + 2\Omega\tanh(\Omega\xi)}\right),$$

where $\Omega = \frac{1}{2}\sqrt{p^2 - 4qr}$, $\xi = x - \frac{1}{2}b_1^2 t$ and b_1, p, q, r are arbitrary constants.

$$u_{14} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr - 3p\left(\frac{2\Omega^2 \operatorname{csch}^2(\Omega\xi)}{p + 2\Delta\coth(\Omega\xi)}\right) - 3\left(\frac{2\Omega^2 \operatorname{csch}^2(\Omega\xi)}{p + 2\Delta\coth(\Omega\xi)}\right)^2,$$

$$v_{14} = -\frac{1}{2}b_1 p - b_1\left(\frac{2\Omega^2 \operatorname{csch}^2(\Omega\xi)}{p + 2\Delta\coth(\Omega\xi)}\right),$$

$$u_{15} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr + 3p\left(\frac{4\Omega^2 \operatorname{sech}(2\Omega\xi)(1 \mp i\sinh(2\Omega\xi))}{p\cosh(2\Omega\xi) + 2\Delta\sinh(2\Omega\xi) \pm i2\Omega}\right)$$
$$-3\left(\frac{4\Omega^2 \operatorname{sech}(2\Omega\xi)(1 \mp i\sinh(2\Omega\xi))}{p\cosh(2\Omega\xi) + 2\Delta\sinh(2\Omega\xi) \pm i2\Omega}\right)^2,$$

$$v_{15} = -\frac{1}{2}b_1 p + b_1\left(\frac{4\Omega^2 \operatorname{sech}(2\Omega\xi)(1 \mp i\sinh(2\Omega\xi))}{p\cosh(2\Omega\xi) + 2\Delta\sinh(2\Omega\xi) \pm i2\Omega}\right),$$

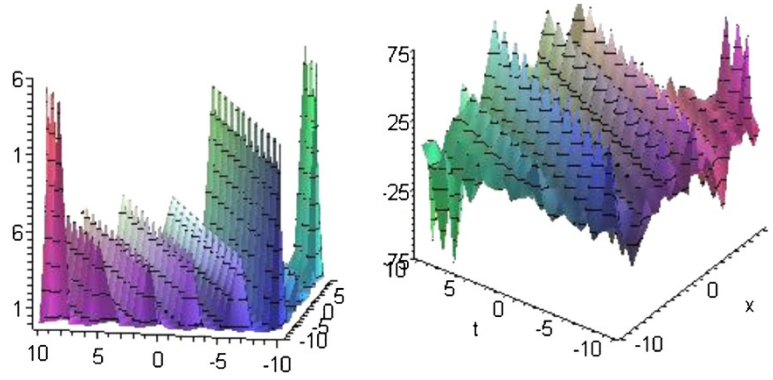

Figure 1 Solitons corresponding to solutions u_1 and v_1 for p=q=2, r=3 and b_1=1.

$$u_{16} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr - 3p\left(\frac{4\Omega^2\,\mathrm{csc}h(2\Omega\xi)\,(1 \pm \cosh(2\Omega\xi))}{p\sinh(2\Omega\xi) + 2\Omega\cosh(2\Delta\xi) \pm 2\Omega}\right) - 3\left(\frac{4\Omega^2\,\mathrm{csc}h(2\Omega\xi)\,(1 \pm \cosh(2\Omega\xi))}{p\sinh(2\Omega\xi) + 2\Omega\cosh(2\Delta\xi) \pm 2\Omega}\right)^2,$$

$$v_{16} = -\frac{1}{2}b_1 p - b_1\left(\frac{4\Omega^2\,\mathrm{csc}h(2\Omega\xi)\,(1 \pm \cosh(2\Omega\xi))}{p\sinh(2\Omega\xi) + 2\Omega\cosh(2\Delta\xi) \pm 2\Omega}\right),$$

$$u_{17} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr - 3p\left(\frac{\Omega^2\,\mathrm{sec}h^2(\Omega\xi/2)}{2\left\{\cosh^2(\Omega\xi/2) - 1\right\}\left\{p + \Omega\left(\tanh(\Omega\xi/2) + \coth(\Omega\xi/2)\right)\right\}}\right) - 3\left(\frac{\Omega^2\,\mathrm{sec}h^2(\Omega\xi/2)}{2\left\{\cosh^2(\Omega\xi/2) - 1\right\}\left\{p + \Omega\left(\tanh(\Omega\xi/2) + \coth(\Omega\xi/2)\right)\right\}}\right)^2,$$

$$v_{17} = -\frac{1}{2}b_1 p - b_1\left(\frac{\Omega^2\,\mathrm{sec}h^2(\Omega\xi/2)}{2\left\{\cosh^2(\Omega\xi/2) - 1\right\}\left\{p + \Omega\left(\tanh(\Omega\xi/2) + \coth(\Omega\xi/2)\right)\right\}}\right),$$

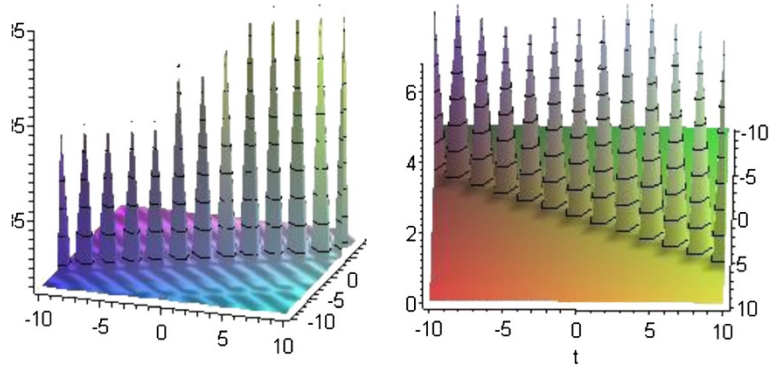

Figure 2 Solitons corresponding to solutions u_5 and v_5 for p=q=1, r=2 and b_1=1.

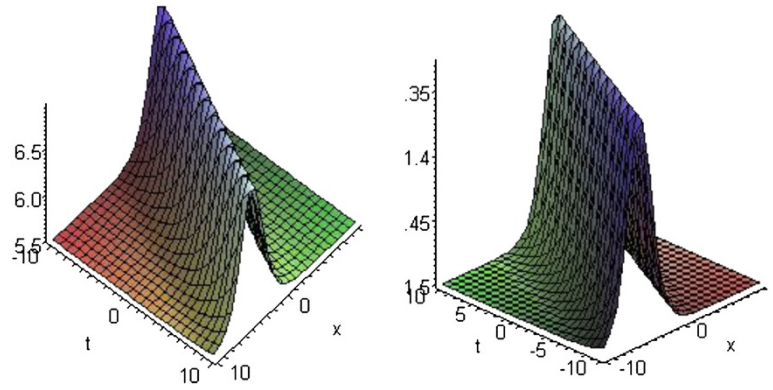

Figure 3 Solitons corresponding to solutions u_{13} and v_{13} for p=3, q=2, r=1 and b_1=1.

$$u_{18} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr - 3p\left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) - \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{(A\sin(2\Omega\xi)+B)\left(pA\sinh(2\Omega\xi)+pB - 2\Omega\sqrt{A^2+B^2}+2A\Omega\cosh(2\Omega\xi)\right)}\right)$$

$$-3\left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) - \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{(A\sin(2\Omega\xi)+B)\left(pA\sinh(2\Omega\xi)+pB - 2\Omega\sqrt{A^2+B^2}+2A\Omega\cosh(2\Omega\xi)\right)}\right)^2,$$

$$v_{18} = -\frac{1}{2}b_1 p - b_1\left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) - \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{(A\sin(2\Omega\xi)+B)\left(pA\sinh(2\Omega\xi)+pB - 2\Omega\sqrt{A^2+B^2}+2A\Omega\cosh(2\Omega\xi)\right)}\right),$$

$$u_{19} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr - 3p\left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) + \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{(A\sin(2\Omega\xi)+B)\left(pA\sinh(2\Omega\xi)+pB + 2\Omega\sqrt{A^2+B^2}+2A\Omega\cosh(2\Omega\xi)\right)}\right)$$

$$-3\left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) + \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{(A\sin(2\Omega\xi)+B)\left(pA\sinh(2\Omega\xi)+pB + 2\Omega\sqrt{A^2+B^2}+2A\Omega\cosh(2\Omega\xi)\right)}\right)^2,$$

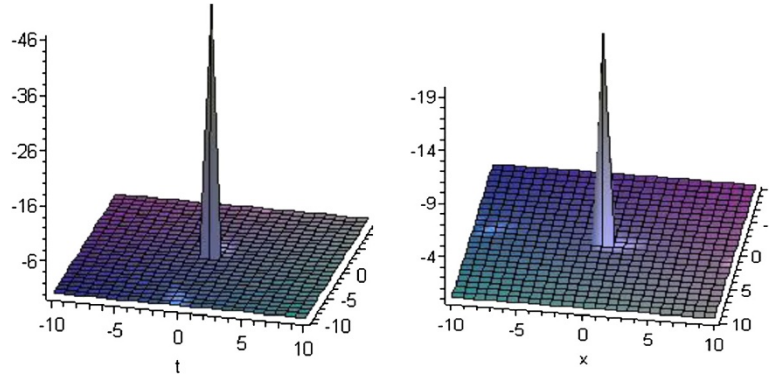

Figure 4 Solitons corresponding to solutions u_{24} and v_{24} for p=3, q=2, r=1 and b_1=5.

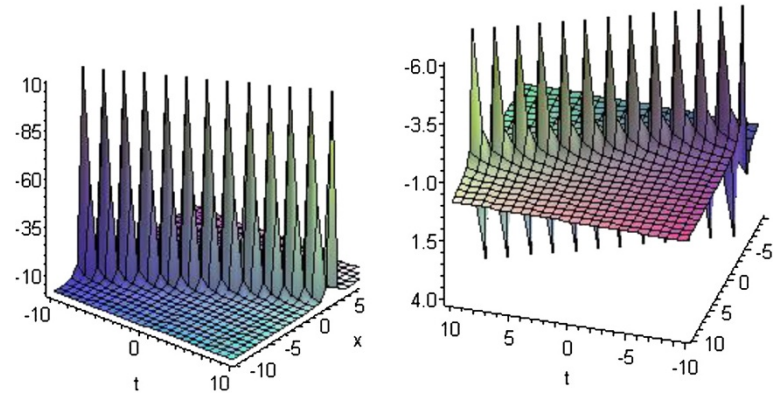

Figure 5 Solitons corresponding to solutions u_{27} and v_{27} for p=0, q=1, r=0 and b_1=5 and c_1=1.

$$v_{19} = -\frac{1}{2}b_1 p - b_1\left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) + \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{(A\sin(2\Omega\xi)+B)\left(pA\sinh(2\Omega\xi)+pB+2\Omega\sqrt{A^2+B^2}+2A\Omega\cosh(2\Omega\xi)\right)}\right),$$

where A and B are two non-zero real constants and satisfies the condition $B^2 - A^2 > 0$.

$$u_{20} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr - 3p\left(\frac{2\Omega^2\,\mathrm{sech}(\Omega\xi)}{2\Omega\sinh(\Omega\xi) - p\cosh(\Omega\xi)}\right) - 3\left(\frac{2\Omega^2\,\mathrm{sech}(\Omega\xi)}{2\Omega\sinh(\Omega\xi) - p\cosh(\Omega\xi)}\right)^2,$$

$$v_{20} = -\frac{1}{2}b_1 p - b_1\left(\frac{2\Omega^2\,\mathrm{sech}(\Omega\xi)}{2\Omega\sinh(\Omega\xi) - p\cosh(\Omega\xi)}\right),$$

$$u_{21} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr + 3p\left(\frac{2\Omega^2\,\mathrm{csch}(\Omega\xi)}{2\Omega\cosh(\Omega\xi) - p\sinh(\Omega\xi)}\right) - 3\left(\frac{2\Omega^2\,\mathrm{csch}(\Omega\xi)}{2\Omega\cosh(\Omega\xi) - p\sinh(\Omega\xi)}\right)^2,$$

$$v_{21} = -\frac{1}{2}b_1 p + b_1\left(\frac{2\Omega^2\,\mathrm{csch}(\Omega\xi)}{2\Omega\cosh(\Omega\xi) - p\sinh(\Omega\xi)}\right),$$

$$u_{22} = -\frac{p^2}{4} - \frac{b_1^2}{4} + 4qr + 3p\left(\frac{4\Omega^2\,\mathrm{sech}(2\Omega\xi)\,(1 \mp i\sinh(2\Omega\xi))}{p\cosh(2\Omega\xi) - 2\Omega\sinh(2\Omega\xi) \mp i2\Omega}\right) - 3\left(\frac{4\Omega^2\,\mathrm{sech}(2\Omega\xi)\,(1 \mp i\sinh(2\Omega\xi))}{p\cosh(2\Omega\xi) - 2\Omega\sinh(2\Omega\xi) \mp i2\Omega}\right)^2,$$

$$v_{22} = -\frac{1}{2}b_1 p + b_1\left(\frac{4\Omega^2\,\mathrm{sec}h(2\Omega\xi)\,(1\mp i\sinh(2\Omega\xi))}{p\cosh(2\Omega\xi)-2\Omega\sinh(2\Omega\xi)\mp i2\Omega}\right),$$

$$u_{23} = -\frac{p^2}{4}-\frac{b_1^2}{4}+4qr+3p\left(\frac{4\Omega^2\,\mathrm{csc}h(2\Omega\xi)\,(1\pm\cosh(2\Omega\xi))}{2\Omega\cosh(2\Omega\xi)-p\sinh(2\Omega\xi)\pm 2\Omega}\right)$$
$$-3\left(\frac{4\Omega^2\,\mathrm{csc}h(2\Omega\xi)\,(1\pm\cosh(2\Omega\xi))}{2\Omega\cosh(2\Omega\xi)-p\sinh(2\Omega\xi)\pm 2\Omega}\right)^2,$$

$$v_{23} = -\frac{1}{2}b_1 p + b_1\left(\frac{4\Omega^2\,\mathrm{csc}h(2\Omega\xi)\,(1\pm\cosh(2\Omega\xi))}{2\Omega\cosh(2\Omega\xi)-p\sinh(2\Omega\xi)\pm 2\Omega}\right),$$

$$u_{24} = -\frac{p^2}{4}-\frac{b_1^2}{4}+4qr+3p\left(\frac{2\Omega^2\,\mathrm{csc}h(\Omega\xi)}{2\Omega\cosh(\Omega\xi)-p\sinh(\Omega\xi)}\right)$$
$$-3\left(\frac{2\Omega^2\,\mathrm{csc}h(\Omega\xi)}{2\Omega\cosh(\Omega\xi)-p\sinh(\Omega\xi)}\right)^2,$$

$$v_{24} = -\frac{1}{2}b_1 p + b_1\left(\frac{2\Omega^2\,\mathrm{csc}h(\Omega\xi)}{2\Omega\cosh(\Omega\xi)-p\sinh(\Omega\xi)}\right).$$

Family 3

When r=0 and $pq\neq 0$, the solutions of Equations (6) and (7) are:

$$u_{25} = -\frac{p^2}{4}-\frac{b_1^2}{4}+4qr+3p\left(\frac{p\,(\cosh(p\xi)-\sinh(p\xi))}{d+\cosh(p\xi)-\sinh(p\xi)}\right)$$
$$-3\left(\frac{p\,(\cosh(p\xi)-\sinh(p\xi))}{d+\cosh(p\xi)-\sinh(p\xi)}\right)^2,$$

$$v_{25} = -\frac{1}{2}b_1 p + b_1\left(\frac{p\,(\cosh(p\xi)-\sinh(p\xi))}{d+\cosh(p\xi)-\sinh(p\xi)}\right),$$

$$u_{26} = -\frac{p^2}{4}-\frac{b_1^2}{4}+4qr+3p\left(\frac{pd}{d+\cosh(p\xi)+\sinh(p\xi)}\right)$$
$$-3\left(\frac{pd}{d+\cosh(p\xi)+\sinh(p\xi)}\right)^2,$$

$$v_{26} = -\frac{1}{2}b_1 p + b_1\left(\frac{pd}{d+\cosh(p\xi)+\sinh(p\xi)}\right).$$

Family 4

When $q\neq 0$ and $r=p=0$, the solutions of Equations (6) and (7) are:

$$u_{27} = -\frac{p^2}{4}-\frac{b_1^2}{4}+4qr-3p\left(\frac{q}{q\xi+c_1}\right)-3\left(\frac{q}{q\xi+c_1}\right)^2,$$
$$v_{27} = -\frac{1}{2}b_1 p - b_1\left(\frac{q}{q\xi+c_1}\right)$$

where c_1 is an arbitrary constant.

Because of the arbitrariness of the parameters b_1, p, q and r in the above families of solution, the physical quantities u and v might possess physically significant rich structures.

Graphical presentation

Graph is a powerful tool for communication and describes lucidly the solutions of the problems. Therefore, some graphs of the solutions are given below. The graphs readily have shown the solitary wave form of the solutions (Figures 1, 2, 3, 4 and 5).

Conclusion

In this article, the alternative (G'/G)-expansion method has been modified by introducing the generalized Riccati equation mapping and obtain abundant exact traveling wave solutions of the (1+1)-dimensional DSW equation with the help of symbolic computation. It is important to point out that the obtained solutions have not been reported in the previous literature. The new type of traveling wave solutions found in this article might have significant impact on future research. We assured the correctness of our solutions by putting them back into the original Equations (6) and (7). This article is only an imploring work and we look forward the modified alternative (G'/G)-expansion method may be applicable to other kinds of NLEEs in mathematical physics. The extension of the method proposed in this paper to solve NLEEs with variable coefficients deserves further investigations.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors, viz. MAA NHMA and STM, with the consultation of each other carried out this work and drafted the manuscript together. All the authors read and approved the final manuscript.

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