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Solving Cauchy reaction-diffusion equation by using Picard method

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Abstract

In this paper, Picard method is proposed to solve the Cauchy reaction-diffusion equation with fuzzy initial condition under generalized H -differentiability. The existence and uniqueness of the solution and convergence of the proposed method are proved in details. Some examples are investigated to verify convergence results and to illustrate the efficiency of the method. Also, we obtain the switching points in examples.

Keywords: Cauchy reaction-diffusion equation; Fuzzy number; Fuzzy-valued function; h -difference; Generalized differentiability; Picard method

Introduction

As we know the fuzzy differential equations FDE are one of the important part of the fuzzy analysis theory that play major role in numerical analysis. For example, population models (Guo et al. 2003), the golden mean (Datta 2003), quantum optics and gravity (El Naschie 2005), control chaotic systems (Feng and Chen 2005; Jiang 2005), medicine (Abbod et al. 2001; Barro and Marin, 2002). Recently, some mathematicians have studied FDE (Abbasbandy and Allahviranloo 2000; Abbasbandy et al. 2004; Abbasbandy et al. 2005; Allahviranloo et al. 2007; Bede 2008; Bede and Gal 2005; Bede et al. 2007; Buckley and Feuring 2000; Buckley and Jowers 2006; Buckley et al. 2002; Chalco-Cano and Romn-Flores 2006; Chalco-Cano and Romn-Flores et al. 2007; Chapko and Johansson 2012; Chen and Ho 1999; Cho and Lan 2007; Congxin and Shiji 1993; Diamond 1999; Diamond 2002; Ding et al. 1997; Dubois 1982; Dou and Hon 2012; Fard 2009a,b; Fard and Bidgoli 2010; Fard and Kamyad 2011; Fard et al. 2009; Fei 2007; Jang et al. 2000; Jowers et al. 2007; Kaleva 1987,1990,2006; Lopez 2008; Ma et al. 1999; Mizukoshi et al. 2007; Oberguggenberger and Pittschmann 1999; Papaschinopoulos 2007; Puri and Ralescu 1983; Seikkala 1987; Solaymani Fard and Ghal-Eh 2011; Song et al. 2000). The fuzzy partial differential equations $FPDE$ are very important in mathematical models of physical, chemical,

biological, economics and other fields. Some mathematicians have studied solution of $FPDE$ by numerical methods (Afshar Kermani and Saburi 2007; Allahviranloo 2002; Barkhordari Ahmadi and Kiani 2011; Buckley and Feuring 1999; Chen et al. 2009; Farajzadeh et al. 2010; Moghadam and Jalal 2011; Rouhparvar et al. 2010; Verma et al. 2009). In this work, we present the Picard method to solve the Cauchy reaction-diffusion equation as follows:

$$\begin{aligned} \tilde{u}_t(x, t) &= \tilde{u}_{xx}(x, t) + \tilde{u}(x, t), & 0 \leq t \leq T, \\ & & a \leq x \leq b, a, b, T \in R. \end{aligned} \quad (1)$$

With fuzzy initial condition:

$$\tilde{u}(x, 0) = \tilde{f}(x). \quad (2)$$

The structure of this paper is organized as follows: In section "Basic concepts", some basic notations and definitions in fuzzy calculus are brought. In section "Description of the method", Eqs.(1,2) are solved by Picard method. The existence and uniqueness of the solution and convergence of the proposed method are proved in section "Existence and convergence analysis" respectively. Finally, in section "Numerical examples", the accuracy of method by solving some numerical examples are illustrated and a brief conclusion is given in section "Conclusion".

Basic concepts

Here basic definitions of a fuzzy number are given as follows, (Allahviranloo 2005; Dubois and Prade 2005; Kauffman and Gupta 1991; Nguyen 1978; Zadeh 1965)

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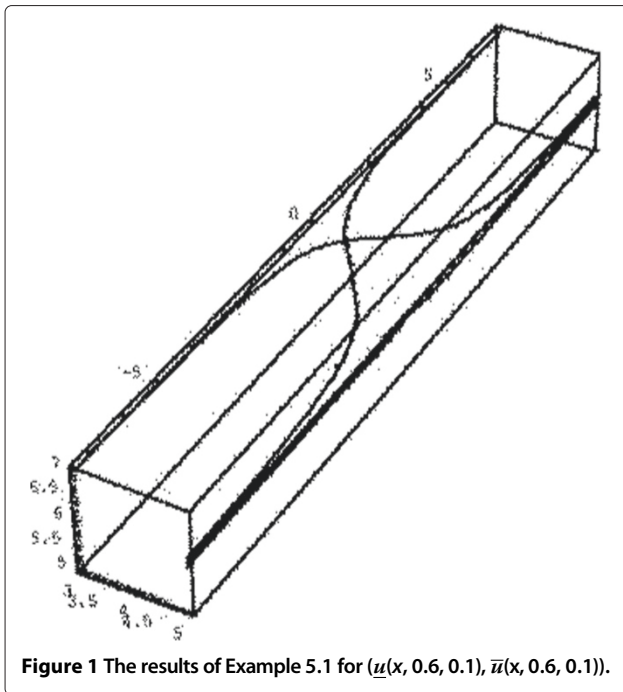


Figure 1 The results of Example 5.1 for $\underline{u}(x, 0.6, 0.1)$, $\bar{u}(x, 0.6, 0.1)$.

Definition 2.1. An arbitrary fuzzy number \tilde{u} in the parametric form is represented by an ordered pair of functions (\underline{u}, \bar{u}) which satisfy the following requirements:

- (i) $\bar{u} : r \rightarrow \bar{u}(r) \in R$ is a bounded left-continuous non-decreasing function over $[0, 1]$,
- (ii) $\underline{u} : r \rightarrow \underline{u}(r) \in R$ is a bounded left-continuous non-increasing function over $[0, 1]$,
- (iii) $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

Definition 2.2. For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$, we use the distance (Hausdorff metric) (Goetschel and Voxman 1986) $D(u(r), v(r)) = \max\{\sup_{r \in [0,1]} |\underline{u}(r) - \underline{v}(r)|, \sup |\bar{u}(r) - \bar{v}(r)|\}$, and it is shown (Puri and Ralescu 1986) that (E^1, D) is a complete metric space and the following properties are well known:

$$\begin{aligned} D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) &= D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v} \in E^1, \\ D(k\tilde{u}, k\tilde{v}) &= |k| D(\tilde{u}, \tilde{v}), \forall k \in R, \tilde{u}, \tilde{v} \in E^1, \\ D(\tilde{u} + \tilde{v}, \tilde{w} + \tilde{e}) &\leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e}), \forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in E^1. \end{aligned}$$

Definition 2.3. Consider $\tilde{x}, \tilde{y} \in E$. If there exists $\tilde{z} \in E$ such that $\tilde{x} = \tilde{y} + \tilde{z}$ then \tilde{z} is called the H - difference of \tilde{x} and \tilde{y} , and is denoted by $\tilde{x} \ominus \tilde{y}$ (Bede and Gal 2005).

Proposition 1. If $\tilde{f} : (a, b) \rightarrow E$ is a continuous fuzzy-valued function then $g(x) = \int_a^x f(t) dt$ is differentiable, with derivative $g'(x) = f(x)$ (Bede and Gal 2005).

Definition 2.4. (see (Bede and Gal 2005)) Let $\tilde{f} : (a, b) \rightarrow E$ and $x_0 \in (a, b)$. We say that \tilde{f} is generalized differentiable at x_0 (Bede-Gal differentiability), if there exists an element $f'(x_0) \in E$, such that:

- i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

or

- ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

- iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

- iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:

$$\lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

Definition 2.5. Let $\tilde{f} : (a, b) \rightarrow E$. We say \tilde{f} is (i)-differentiable on (a, b) if \tilde{f} is differentiable in the sense (i) of Definition (2.7) and similarly for (ii), (iii) and (iv) differentiability.

Definition 2.6. (see (Chalco-Cano and Romn-Flores 2006)) Let $\tilde{f} : (a, b) \rightarrow E$ and $x_0 \in (a, b)$. A point $x_0 \in (a, b)$ is said to be a switching point for the differentiability of \tilde{f} , if in any neighborhood V of x_0 there exist points $x_1 < x_0 < x_2$ such that

(type 1) \tilde{f} is differentiable at x_1 in the sense (i) of Definition (2.6) while it is not differentiable in the sense (ii) of Definition (2.6), and \tilde{f} is differentiable at x_2 in the sense (ii) of Definition (2.6) while it is not differentiable in the sense (i) or Definition (2.6), or

(type 2) \tilde{f} is differentiable at x_1 in the sense (ii) of Definition (2.6) while it is not differentiable in the sense (i) of Definition (2.6), and \tilde{f} is differentiable at x_2 in the sense (i) of Definition (2.6) while it is not differentiable in the sense (ii) or Definition (2.6).

Proposition 2. (see (Chalco-Cano and Romn-Flores 2006)) Let $\tilde{f} : (a, b) \rightarrow E$ and $x_0 \in (a, b)$.

- (a) If $x_0 \in (a, b)$ is a switching point for the differentiability of \tilde{f} of type 1, then \tilde{f} is differentiable at x_0 in the form (iv).
- (b) If $x_0 \in (a, b)$ is a switching point for the differentiability of \tilde{f} of type 2, then \tilde{f} is differentiable at x_0 in the form (iii).

Definition 2.7. A triangular fuzzy number is defined as a fuzzy set in E^1 , that is specified by an ordered triple $u = (a, b, c) \in R^3$ with $a \leq b \leq c$ such that $u(r) = [\underline{u}(r), \bar{u}(r)]$ are the endpoints of r -level sets for all $r \in [0, 1]$, where $\underline{u}(r) = a + (b - a)r$ and $\bar{u}(r) = c - (c - b)r$. Here, $\underline{u}(0) = a, \bar{u}(0) = c, \underline{u}(1) = \bar{u}(1) = b$, which is denoted by $u(1)$. The set of triangular fuzzy numbers will be denoted by E^1 .

Definition 2.8. (see (Chalco-Cano and Romn-Flores 2006)) The mapping $\tilde{f} : T \rightarrow E^n$ for some interval T is called a fuzzy process. Therefore, its r -level set can be written as follows:

$$(f(t))(r) = [\underline{f}(t, r), \bar{f}(t, r)], \quad t \in T, \quad r \in [0, 1].$$

Definition 2.9. (see (Chalco-Cano and Romn-Flores 2006)) Let $f : T \rightarrow E^n$ be Hukuhara differentiable and denote $(f(t))(r) = [\underline{f}(t, r), \bar{f}(t, r)]$. Then, the boundary function $\underline{f}(t, r)$ and $\bar{f}(t, r)$ are differentiable (or Seikkala differentiable) and

$$(f'(t))(r) = [\underline{f}'(t, r), \bar{f}'(t, r)], \quad t \in T, \quad r \in [0, 1].$$

If f is (ii)-differentiable then

$$(f'(t))(r) = [\bar{f}'(t, r), \underline{f}'(t, r)], \quad t \in T, \quad r \in [0, 1].$$

Description of the method

To obtain the approximation solution of Eqs.(1,2), based on Definition (2.6) we have two cases as follows:

Case (1): $\tilde{u}(x, t)$ is (i)-differentiable, in this case we have,

$$\tilde{u}(x, t) = \tilde{f}(x) + \int_0^t [\tilde{u}(x, t) + \tilde{u}_{xx}(x, t)] dt. \quad (3)$$

Case (2): $\tilde{u}(x, t)$ is (ii)-differentiable, in this case we have,

$$\tilde{u}(x, t) = \tilde{f}(x) \ominus (-1) \cdot \int_0^t [\tilde{u}(x, t) + \tilde{u}_{xx}(x, t)] dt. \quad (4)$$

Now, we can write successive iterations (by using Picard method) as follows:

Case (1):

$$\begin{aligned} \tilde{u}_0(x, t) &= \tilde{f}(x), \\ \tilde{u}_{n+1}(x, t) &= \tilde{f}(x) + \int_0^t [\tilde{u}_n(x, t) + \tilde{u}_{n,xx}(x, t)] dt, \quad n \geq 0. \end{aligned} \quad (5)$$

Case (2):

$$\begin{aligned} \tilde{u}_0(x, t) &= \tilde{f}(x), \\ \tilde{u}_{n+1}(x, t) &= \tilde{f}(x) \ominus (-1) \cdot \int_0^t [\tilde{u}_n(x, t) + \tilde{u}_{n,xx}(x, t)] dt, \quad n \geq 0. \end{aligned} \quad (6)$$

Remark 1. For \tilde{u}_{xx} we have cases as follows:

Case (1): \tilde{u} and \tilde{u}' be (i)-differentiable and \tilde{u} and \tilde{u}' be (ii)-differentiable

$$\tilde{u}_{xx}(x, t) = [\underline{u}_{xx}(x, t, r), \bar{u}_{xx}(x, t, r)].$$

Case (2): \tilde{u} is (i)-differentiable and \tilde{u}' is (ii)-differentiable and \tilde{u} is (ii)-differentiable and \tilde{u}' is (i)-differentiable

$$\tilde{u}''(x, t) = [\underline{u}''(x, t, r), \bar{u}''(x, t, r)].$$

Remark 2. We discuss about switching points as follows:

Case (1): \tilde{u} is (i)-differentiable

If $\frac{\partial \underline{u}(x,t,r)}{\partial x} < 0, \frac{\partial \bar{u}(x,t,r)}{\partial x} > 0, x \in [a, x_0]$ and $\frac{\partial \underline{u}(x,t,r)}{\partial x} > 0, \frac{\partial \bar{u}(x,t,r)}{\partial x} < 0, x \in (x_0, b]$ and $r \in [0, 1]$ then we have $\tilde{u}'(x, t) = [\underline{u}'(x, t, r), \bar{u}'(x, t, r)]$ and x_0 is a switching point in the form (iv).

If $\frac{\partial^2 \underline{u}(x,t,r)}{\partial x^2} < 0, \frac{\partial^2 \bar{u}(x,t,r)}{\partial x^2} > 0, x \in [a, x_1]$ and $\frac{\partial^2 \underline{u}(x,t,r)}{\partial x^2} > 0, \frac{\partial^2 \bar{u}(x,t,r)}{\partial x^2} < 0, x \in (x_1, b]$ and $r \in [0, 1]$ then we have $\tilde{u}''(x, t) = [\underline{u}''(x, t, r), \bar{u}''(x, t, r)]$ and x_1 is a switching point in the form (iv).

Case (2): \tilde{u} is (ii)-differentiable

If $\frac{\partial \underline{u}(x,t,r)}{\partial x} > 0, \frac{\partial \bar{u}(x,t,r)}{\partial x} < 0, x \in [a, x_0]$ and $\frac{\partial \underline{u}(x,t,r)}{\partial x} < 0, \frac{\partial \bar{u}(x,t,r)}{\partial x} > 0, x \in (x_0, b]$ and $r \in [0, 1]$ then we have $\tilde{u}'(x, t) = [\underline{u}'(x, t, r), \bar{u}'(x, t, r)]$ and x_0 is a switching point in the form (iii).

If $\frac{\partial^2 \underline{u}(x,t,r)}{\partial x^2} > 0, \frac{\partial^2 \bar{u}(x,t,r)}{\partial x^2} < 0, x \in [a, x_1]$ and $\frac{\partial^2 \underline{u}(x,t,r)}{\partial x^2} < 0, \frac{\partial^2 \bar{u}(x,t,r)}{\partial x^2} > 0, x \in (x_1, b]$ and $r \in [0, 1]$ then we have $\tilde{u}''(x, t) = [\underline{u}''(x, t, r), \bar{u}''(x, t, r)]$ and x_1 is a switching point in the form (iii).

Case (3): \tilde{u} is (i)-differentiable

If $\frac{\partial \underline{u}(x,t,r)}{\partial x} < 0, \frac{\partial \bar{u}(x,t,r)}{\partial x} > 0, \forall x \in [a, b]$ then we have $\tilde{u}'(x, t) = [\underline{u}'(x, t, r), \bar{u}'(x, t, r)]$.
 If $\frac{\partial^2 \underline{u}(x,t,r)}{\partial x^2} < 0, \frac{\partial^2 \bar{u}(x,t,r)}{\partial x^2} > 0, x \in [a, x_0]$ and $\frac{\partial^2 \underline{u}(x,t,r)}{\partial x^2} > 0, \frac{\partial^2 \bar{u}(x,t,r)}{\partial x^2} < 0, x \in (x_0, b]$ and $r \in [0, 1]$

then we have $\tilde{u}''(x, t) = [\underline{u}''(x, t, r), \overline{u}''(x, t, r)]$ and x_0 is a switching point in the form (iv).

Case (4): \tilde{u} is (ii)-differentiable

If $\frac{\partial \underline{u}(x, t, r)}{\partial x} > 0, \frac{\partial \overline{u}(x, t, r)}{\partial x} < 0, \forall x \in [a, b]$ then we have

$\tilde{u}'(x, t) = [\underline{u}'(x, t, r), \overline{u}'(x, t, r)]$.

If $\frac{\partial^2 \underline{u}(x, t, r)}{\partial x^2} > 0, \frac{\partial^2 \overline{u}(x, t, r)}{\partial x^2} < 0, x \in [a, x_1]$ and

$\frac{\partial^2 \underline{u}(x, t, r)}{\partial x^2} < 0, \frac{\partial^2 \overline{u}(x, t, r)}{\partial x^2} > 0, x \in (x_1, b]$ and $r \in [0, 1]$

then we have $\tilde{u}''(x, t) = [\underline{u}''(x, t, r), \overline{u}''(x, t, r)]$ and x_1 is a switching point in the form (iii).

Existence and convergence analysis

In this section we are going to prove the existence and uniqueness of the solution and convergence of the method by using the following assumptions.

Consider $\tilde{f}(x)$ is bounded for all $x \in [a, b]$ and

$$D(\tilde{u}_{xx}(x, t), \tilde{u}_{xx}^*(x, t)) \leq LD(\tilde{u}(x, t), \tilde{u}^*(x, t)), \quad L > 0.$$

Let,

$$\alpha = (T + TL).$$

Lemma 1. If $\tilde{u}, \tilde{v}, \tilde{w} \in E^n$ and $\lambda \in R$, then,

- (i) $D(\tilde{u} \ominus \tilde{v}, \tilde{u} \ominus \tilde{w}) = D(\tilde{v}, \tilde{w})$,
- (ii) $D(\ominus \lambda \tilde{u}, \ominus \lambda \tilde{v}) = |\lambda| D(\tilde{u}, \tilde{v})$.

Proof (i). By the definition of D , we have,

$$\begin{aligned} & D(\tilde{u} \ominus \tilde{v}, \tilde{u} \ominus \tilde{w}) \\ &= \max\{\sup_{r \in [0,1]} | \underline{u}(r) - \underline{v}(r) - \underline{u}(r) - \underline{w}(r) |, \\ & \quad \sup_{r \in [0,1]} | \overline{u}(r) - \overline{v}(r) - \overline{u}(r) - \overline{w}(r) | \} \\ &= \max\{\sup_{r \in [0,1]} | \underline{u}(r) - \underline{v}(r) - (\underline{u}(r) - \underline{w}(r)) |, \\ & \quad \sup_{r \in [0,1]} | \overline{u}(r) - \overline{v}(r) - (\overline{u}(r) - \overline{w}(r)) | \} \\ &= \max\{\sup_{r \in [0,1]} | \underline{w}(r) - \underline{v}(r) |, \sup_{r \in [0,1]} | \overline{w}(r) - \overline{v}(r) | \} \\ &= \max\{\sup_{r \in [0,1]} | \underline{v}(r) - \underline{w}(r) |, \sup_{r \in [0,1]} | \overline{v}(r) - \overline{w}(r) | \} \\ &= D(\tilde{v}, \tilde{w}). \quad \square \end{aligned}$$

Proof (ii):

$$\begin{aligned} & D(\ominus \lambda \tilde{u}, \ominus \lambda \tilde{v}) \\ &= \max\{\sup_{r \in [0,1]} | \lambda \underline{u}(r) - \lambda \underline{v}(r) |, \\ & \quad \sup_{r \in [0,1]} | \lambda \overline{u}(r) - \lambda \overline{v}(r) | \} \\ &= \max\{\sup_{r \in [0,1]} | \lambda \underline{u}(r) - \lambda \underline{v}(r) |, \\ & \quad \sup_{r \in [0,1]} | \lambda \overline{u}(r) - \lambda \overline{v}(r) | \} \\ &= D(\lambda \tilde{u}, \lambda \tilde{v}) = |\lambda| D(\tilde{u}, \tilde{v}). \quad \square \end{aligned}$$

Table 1 Numerical results for Example 5.1

x	$(\underline{u}, r = 0.1, n = 25, t = 0.6)$	$(\overline{u}, r = 0.1, n = 25, t = 0.6)$
-0.2	0.34245	0.68532
-0.1	0.35571	0.67259
0.0	0.36438	0.66135
0.1	0.35724	0.67056
0.2	0.34862	0.67843
0.3	0.33597	0.68453
0.4	0.32551	0.69064

Theorem 1. Let $0 < \alpha < 1$, then Eqs.(1,2), have an unique solution and the solution $\tilde{u}_n(x, t)$ obtained from the relation (8) using Picard method converges to the exact solution of the problems (1,2) when \tilde{u} is (ii)-differentiable.

Proof. Let \tilde{u} and \tilde{u}^* be two different solutions of Eqs.(1,2) then

$$\begin{aligned} & D(\tilde{u}(x, t), \tilde{u}^*(x, t)) \\ &= D(\tilde{f}(x) \ominus (-1) \cdot \int_0^t [\tilde{u}(x, t) + \tilde{u}_{xx}(x, t)] dt, \\ & \quad \tilde{f}(x) \ominus (-1) \cdot \int_0^t [\tilde{u}^*(x, t) + \tilde{u}_{xx}^*(x, t)] dt) \\ &= D(\ominus(-1) \cdot \int_0^t [\tilde{u}(x, t) + \tilde{u}_{xx}(x, t)] dt, \\ & \quad \ominus(-1) \cdot \int_0^t [\tilde{u}^*(x, t) + \tilde{u}_{xx}^*(x, t)] dt) \\ &\leq T(D(\tilde{u}(x, t), \tilde{u}^*(x, t))) + TL(D(\tilde{u}(x, t), \tilde{u}^*(x, t))) \\ &= \alpha D(\tilde{u}(x, t), \tilde{u}^*(x, t)). \end{aligned}$$

From which we get $(1 - \alpha)D(\tilde{u}(x, t), \tilde{u}^*(x, t)) \leq 0$. Since $0 < \alpha < 1$, then $D(\tilde{u}(x, t), \tilde{u}^*(x, t)) = 0$. Implies $\tilde{u}(x, t) = \tilde{u}^*(x, t)$.

Also, we have

$$D(\tilde{u}_{n+1}(x, t), \tilde{u}(x, t)) \leq \alpha D(\tilde{u}_n, \tilde{u}).$$

Since, $0 < \alpha < 1$, then $D(\tilde{u}_n(x, t), \tilde{u}(x, t)) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $\tilde{u}_n(x, t) \rightarrow \tilde{u}(x, t)$. \square

Table 2 Numerical results for Example 5.2

x	$(\underline{v}, r = 0.2, n = 17, t = 0.7)$	$(\overline{v}, r = 0.2, n = 17, t = 0.7)$
0.1	0.3726508	0.7023467
0.2	0.3827526	0.6939411
0.3	0.3957162	0.6848745
0.4	0.4136805	0.6612719
0.5	0.4372286	0.6559328
0.6	0.4438574	0.6337306

Table 3 Numerical results for Example 5.2

x	$(\underline{u}, r = 0.2, n = 21, t = 0.7)$	$(\bar{v}, r = 0.2, n = 21, t = 0.7)$
0.1	0.4646316	0.7954225
0.2	0.4717726	0.7844835
0.3	0.4823549	0.7737607
0.4	0.5074658	0.7528153
0.5	0.5263437	0.7474326
0.6	0.5309183	0.7215178

Remark 3. The proof of other case is similar to the previous theorems.

Numerical examples

In this section, we solve the Cauchy reaction-diffusion equation by using the Picard method. The program has been provided with Mathematica 6 according to the following algorithm where ϵ is a given positive value.

Algorithm :

- Step 1.** Set $n \leftarrow 0$.
- Step 2.** Calculate the recursive relations (7) or (8).
- Step 3.** If $D(\tilde{u}_{n+1}(x, t), \tilde{u}_n(x, t)) < \epsilon$ then go to step 4, else $n \leftarrow n + 1$ and go to step 2.
- Step 4.** Print $\tilde{u}_n(x, t)$ as the approximate of the exact solution.

Example 5.1. Consider the Cauchy reaction-diffusion equation as follows:

$$\tilde{u}_t(x, t) = \tilde{u}_{xx}(x, t) + \tilde{u}(x, t). \quad (7)$$

With initial condition:

$$\tilde{u}(x, 0) = \tilde{f}(x) = ((1 - r)x^3, (r - 1)x^3). \quad (8)$$

$\epsilon = 10^{-4}$. $x = 0$ is a switching point.

Case (1): $n = 22$ and $\alpha = 0.8652$.

Case (2): $\alpha = 0.84569$.

Table 1 shows that, the approximation solution of the Cauchy reaction-diffusion equation is convergent with 25 iterations by using the Picard method when \tilde{u} is (ii)-differentiable.

Example 5.2. Consider the Cauchy reaction-diffusion equation as follows:

$$\tilde{u}_t(x, t) = \tilde{u}_{xx}(x, t) + \tilde{u}(x, t). \quad (9)$$

With initial condition:

$$\tilde{u}(x, 0) = \tilde{f}(x) = (x^2 + 1, x^2 + 2, x^2 + 3). \quad (10)$$

$\epsilon = 10^{-3}$.

Case (1): $\alpha = 0.7546$.

Table 2 shows that, the approximation solution of the Cauchy reaction-diffusion equation is convergent

with 17 iterations by using the Picard method when \tilde{u} is (i)-differentiable.

Case (2): $\alpha = 0.7762$.

Table 3 shows that, the approximation solution of the Cauchy reaction-diffusion equation is convergent with 21 iterations by using the Picard method when \tilde{u} is (ii)-differentiable.

Conclusion

The Picard method has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which convergent are rapidly to exact solutions. In this work, the Picard method has been successfully employed to obtain the approximate solution of the Cauchy reaction-diffusion equation under generalized H -differentiability.

Received: 21 July 2012 Accepted: 27 September 2012

Published: 13 March 2013

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doi:10.1186/2193-1801-2-108

Cite this article as: Sadigh Behzadi: Solving Cauchy reaction-diffusion equation by using Picard method. *SpringerPlus* 2013 **2**:108.

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