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A study on vague graphs

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Abstract
The main purpose of this paper is to introduce the notion of vague h-morphism on vague graphs and regular vague graphs. The action of vague h-morphism on vague strong regular graphs are studied. Some elegant results on weak and co weak isomorphism are derived. Also, μ-complement of highly irregular vague graphs are defined.

Keywords: Vague graph, Vague h-morphism, Highly irregular vague graph

Background

Gau and Buehrer (1993) proposed the concept of vague set in 1993, by replacing the value of an element in a set with a subinterval of [0, 1]. Namely, a true-membership function $t_v(x)$ and a false membership function $f_v(x)$ are used to describe the boundaries of the membership degree. The initial definition given by Kauffman (1973) of a fuzzy graph was based the fuzzy relation proposed by Zadeh. Later Rosenfeld (1975) introduced the fuzzy analogue of several basic graph-theoretic concepts. After that, Mordeson and Nair (2001) defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. Sunitha and Vijayakumar (2002) studied some properties of complement on fuzzy graphs. Many classifications of fuzzy graphs can be found in Boorzooei et al. (2016a, b), Rashmanlou and Pal (2013a, b), Rashmanlou et al. (2015a, b, c), Pramanik et al. (2016a, b), Samanta and Pal (2011b, 2012b, 2015, 2016) and Samanta et al. (2014c, d). Recently, Samanta and Pal (2011a, 2012a, 2013, 2014a, 2015) and Samanta et al. (2014b) defined different types of fuzzy graphs and established some important properties. To extent the theory of fuzzy graphs, Akram et al. (2014) introduced vague hypergraphs. After that, Ramakrishna (2009) introduced the concept of vague graphs and studied some of their properties.

In this paper, we introduce the notion of vague h-morphism on vague graphs and study the action of vague h-morphism on vague strong regular graphs. We derive some elegant results on weak and co weak isomorphism. Also, we define μ-complement of highly irregular vague graphs.

Preliminaries

By a graph $G^* = (V, E)$, we mean a non-trivial, finite connected and undirected graph without loops or multiple edges. A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ with $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$, for all $u, v \in V$, where V is a finite non-empty set and $\wedge$ denote minimum.

Definition 1 (Gau and Buehrer 1993) A vague set A on a set X is a pair $(t_A; f_A)$ where t_A and f_A are real valued functions defined on $X \to [0, 1]$, such that $t_A(x) + f_A(x) < 1$ for all $x \in X$. The interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A.

In the above definition, $t_A(x)$ is considered as the lower bound for degree of membership of x in A and $f_A(x)$ is the lower bound for negative of membership of x in A. So, the degree of membership of x in the vague set A is characterized by the interval $[t_A(x), 1 - f_A(x)]$.

The definition of intuitionistic fuzzy graphs is follows. Let V be a non-empty set. An *intuitionistic fuzzy set (IFS)* in V is represented by (V, μ, ν), where $\mu : V \to [0, 1]$ and $\nu : V \to [0, 1]$ are membership function and non-membership function respectively such that $0 \leq \mu(x) + \nu(x) \leq 1$ for all $x \in V$. Though intuitionistic fuzzy sets and vague sets look similar, analytically vague sets are more appropriate when representing vague data. The difference between them is discussed below.

The membership interval of an element x for vague set A is $[t_A(x), 1 - f_A(x)]$. But, the membership value for an element y in an intuitionistic fuzzy set B is $\langle \mu_B(y), \nu_B(y) \rangle$. Here, the semantics of t_A is the same as with A and μ_B is the same as with B. However, the boundary is able to indicate the possible existence of a data value. This difference gives rise to a simpler but meaningful graphical view of data sets (see Fig. 1). It can be seen that, the shaded part formed by the boundary in a given Vague Set naturally represents the possible existence of data. Thus, this "hesitation region" corresponds to the intuition of representing vague data. We will see more benefits of using vague membership intervals in capturing data semantics in subsequent sections.

Let X and Y be ordinary finite non-empty sets. We call a vague relation to be a vague subset of $X \times Y$, that is an expression R defined by

$$R = \{\langle (x, y), t_R(x, y), f_R(x, y) \rangle \mid x \in X, y \in Y\}$$

where $t_R : X \times Y \to [0, 1]$, $f_R : X \times Y \to [0, 1]$, which satisfies the condition $0 \leq t_R(x, y) + f_R(x, y) \leq 1$, for all $(x, y) \in X \times Y$. A vague relation R on X is called reflexive if $t_R(x, x) = 1$ and $f_R(x, x) = 0$ for all $x \in X$. A vague relation R is symmetric if $t_R(x, y) = t_R(y, x)$ and $f_R(x, y) = f_R(y, x)$, for all $x, y \in X$.

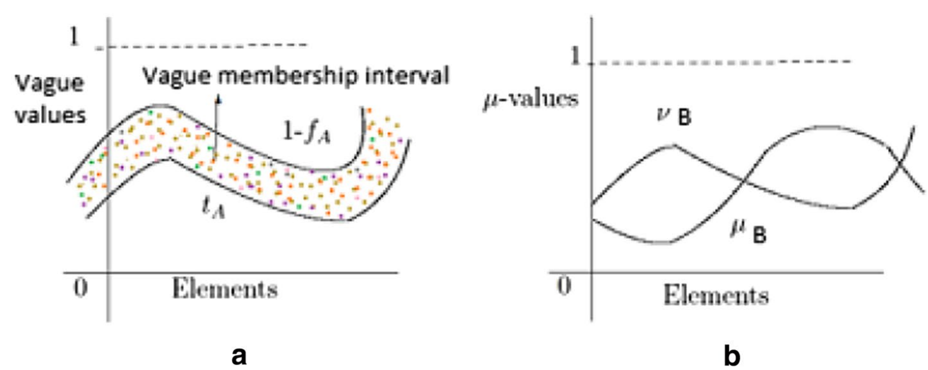

Fig. 1 Comparison between **a** vague sets and **b** intutionistic fuzzy sets

Definition 2 Let $G^* = (V, E)$ be a crisp graph. A pair $G = (A, B)$ is called a vague graph on a crisp graph $G^* = (V, E)$, where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on $E \subseteq V \times V$ such that $t_B(xy) \leq \min(t_A(x), t_A(y))$ and $f_B(xy) \geq \max(f_A(x), f_A(y))$ for each edge $xy \in E$. A vague graph G is called strong if $t_B(xy) = \min(t_A(x), t_A(y))$ and $f_B(xy) = \max(f_A(x), f_A(y))$ for all $x, y \in V$.

Definition 3 The vague graph G is said to be regular if $\sum_{v_j, v_i \neq v_j} t_B(v_i v_j) = \text{constant}$ and $\sum_{v_j, v_i \neq v_j} f_B(v_i v_j) = \text{constant}$, for all $v_i \in V$. Moreover, it is called strong regular if

1. $t_B(v_i v_j) = \min\{t_A(v_i), t_A(v_j)\}$ and $f_B(v_i v_j) = \max\{f_A(v_i), f_A(v_j)\}$.
2. $\sum_{v_j, v_{i \neq v_j}} t_B(v_i v_j) = \text{constant}$ and $\sum_{v_j, v_{i \neq v_j}} f_B(v_i v_j) = \text{constant}$.

Definition 4 The complement of a vague graph $G = (A, B)$ is a vague graph $\overline{G} = (\overline{A}, \overline{B})$ where $\overline{A} = A$ and $\overline{B}$ is described as follows. The true and false membership values for edges of $\overline{G}$ are given below.

$$\overline{t_B}(uv) = t_A(u) \wedge t_A(v) - t_B(uv) \text{ and } \overline{f_B}(uv) = f_B(uv) - f_A(u) \vee f_A(v), \text{ for all } u, v \in V.$$

Definition 5 Let G_1 and G_2 be two vague graphs.

1. A homomorphism h from G_1 to G_2 is a mapping $h : V_1 \to V_2$ which satisfies the following conditions:
   (a) $t_{A_1}(x_1) \leq t_{A_2}(h(x_1)),\ f_{A_1}(x_1) \geq f_{A_2}(h(x_1))$,
   (b) $t_{B_1}(x_1 y_1) \leq t_{B_2}(h(x_1)h(y_1)), f_{B_1}(x_1 y_1) \geq f_{B_2}(h(x_1)h(y_1))$ for all $x_1 \in V_1, x_1 y_1 \in E_1$.
2. An isomorphism h from G_1 to G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:
   (c) $t_{A_1}(x_1) = t_{A_2}(h(x_1)),\ f_{A_1}(x_1) = f_{A_2}(h(x_1))$,
   (d) $t_{B_1}(x_1 y_1) = t_{B_2}(h(x_1)h(y_1)),\ f_{B_1}(x_1 y_1) = f_{B_2}(h(x_1)h(y_1))$, for all $x_1 \in V_1$, $x_1 y_1 \in E_1$.
3. A weak isomorphism h from G_1 to G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:
   (e) h is homomorphism,
   (f) $t_{A_1}(x_1) = t_{A_2}(h(x_1)),\ f_{A_1}(x_1) = f_{A_2}(h(x_1))$ for all $x_1 \in V_1$. Thus a weak isomorphism preserves the weights of the nodes but not necessarily the weights of the arcs.
4. A co weak isomorphism h from G_1 to G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies:
   (g) h is homomorphism,
   (h) $t_{B_1}(x_1 y_1) = t_{B_2}(h(x_1)h(y_1)),\ f_{B_1}(x_1 y_1) = f_{B_2}(h(x_1)h(y_1))$, for all $x_1 y_1 \in E_1$.

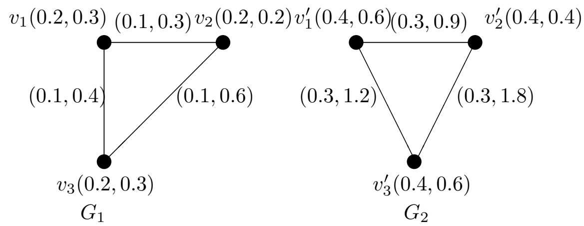

Fig. 2 h-Morphism of vague graphs G_1 and G_2

Definition 6 Let $G = (A, B)$ be a vague graph on G^*.

1. The open degree of a vertex u is defined as $\deg(u) = (d_t(u), d_f(u))$, where

$$d_t(u) = \sum_{\substack{u \neq v \\ v \in V}} t_B(uv) \text{ and}$$

$$d_f(u) = \sum_{\substack{u \neq v \\ v \in V}} f_B(uv).$$

2. The order of G is defined and denoted as

$$O(G) = \left(\sum_{u \in V} t_A(u), \sum_{u \in V} f_A(u) \right).$$

3. The size of G is defined as $S(G) = (S_t(G), S_f(G)) = \left(\sum_{\substack{u \neq v \\ u,v \in V}} t_B(uv), \sum_{\substack{u \neq v \\ u,v \in V}} f_B(uv) \right)$.

Regularity on isomorphic vague graph

In this section, we introduce the notion of vague h-morphism on vague graphs and regular vague graph. We derive some elegant results on weak and co weak isomorphisms.

Definition 7 Let G_1 and G_2 be two vague graphs on (V_1, E_1) and (V_2, E_2), respectively. A bijective function $h : V_1 \to V_2$ is called vague morphism or vague h-morphism if there exists positive numbers k_1 and k_2 such that (i) $t_{A_2}(h(u)) = k_1 t_{A_1}(u)$ and $f_{A_2}(h(u)) = k_1 f_{A_1}(u)$, for all $u \in V_1$, (ii) $t_{B_2}(h(u)h(v)) = k_2 t_{B_1}(uv)$ and $f_{B_2}(h(u)h(v)) = k_2 f_{B_1}(uv)$, for all $uv \in E_1$. In such a case, h will be called a (k_1, k_2) vague h-morphism from G_1 to G_2. If $k_1 = k_2 = k$, we call h, a vague k-morphism. When $k = 1$ we obtain usual vague morphism.

Example 8 Consider two vague graphs G_1 and G_2 defined as follows (see Fig. 2). Here, there is a vague h-morphism such that $h(v_1) = v'_1$, $h(v_2) = v'_2$, $h(v_3) = v'_3$, $k_1 = 2$, and $k_2 = 3$.

Theorem 9 *The relation h-morphism is an equivalence relation in the collection of vague graphs.*

Proof Consider the collection of vague graphs. Define the relation $G_1 \approx G_2$ if there exists a (k_1, k_2) h-morphism from G_1 to G_2 where both K_1 and K_2 are non-zero. Consider the identity morphism from G_1 to G_1. It is a (1, 1) morphism from G_1 to G_1 and hence $\approx$ is reflexive.

Let $G_1 \approx G_2$. Then there exists a (k_1, k_2) morphism from G_1 to G_2, for some non-zero k_1 and k_2. Therefore $t_{A_2}(h(u)) = k_1 t_{A_1}(u)$, $f_{A_2}(h(u)) = k_1 f_{A_1}(u)$, for all $u \in V_1$ and $t_{B_2}(h(u)h(v)) = k_2 t_{B_1}(uv)$ and $f_{B_2}(h(u)h(v)) = k_2 f_{B_1}(uv)$, for all $uv \in E_1$. Consider $h^{-1} : G_2 \to G_1$. Let $x, y \in V_2$. Since h^{-1} is bijective, $x = h(u)$, $y = h(v)$, for some $u, v \in V_2$. Now, $t_{A_1}(h^{-1}(x)) = t_{A_1}(h^{-1}(h(u))) = t_{A_1}(u) = \frac{1}{k_1} t_{A_2}(h(u)) = \frac{1}{k_1} t_{A_2}(x)$. $f_{A_1}(h^{-1}(x)) = f_{A_1}(h^{-1}(h(u))) = f_{A_1}(u) = \frac{1}{k_1} f_{A_2}(h(u)) = \frac{1}{k_1} f_{A_2}(x)$. $t_{B_1}(h^{-1}(x)h^{-1}(y)) = t_{B_1}(h^{-1}(h(u))h^{-1}(h(v))) = t_{B_1}(uv) = \frac{1}{k_2} t_{B_2}(h(u)h(v)) = \frac{1}{k_2} t_{B_2}(xy)$, $t_{B_1}(h^{-1}(x)h^{-1}(y)) = t_{B_1}(h^{-1}(h(u))h^{-1}(h(v))) = t_{B_1}(uv) = \frac{1}{k_2} t_{B_2}(h(u)h(v)) = \frac{1}{k_2} t_{B_2}(xy)$, $f_{B_1}(h^{-1}(x)h^{-1}(y)) = f_{B_1}(h^{-1}(h(u))h^{-1}(h(v))) = f_{B_1}(uv) = \frac{1}{k_2} f_{B_2}(h(u)h(v)) = \frac{1}{k_2} f_{B_2}(xy)$. Thus there exists $\left(\frac{1}{k_1}, \frac{1}{k_2}\right)$ morphism from G_2 to G_1. Therefore, $G_2 \approx G_1$ and hence $\approx$ is symmetric.

Let $G_1 \approx G_2$ and $G_2 \approx G_3$. Then there exists a (k_1, k_2) morphism from G_1 to G_2 say h for some non-zero k_1 and k_2 and there exists (k_3, k_4) morphism from G_2 to G_3 say g for some non-zero k_3 and k_4. So, $t_{A_3}(g(x)) = k_3 t_{A_2}(x)$ and $f_{A_3}(g(x)) = k_3 f_{A_2}(x)$, for all $x \in V_2$ and $t_{B_3}(g(x)g(y)) = k_4 t_{B_2}(xy)$ and $f_{B_3}(g(x)g(y)) = k_4 f_{B_2}(xy)$, for all $xy \in E_2$. Let $f = g \circ h : G_1 \to G_3$. Now,

$$\begin{aligned}
t_{A_3}(f(u)) = t_{A_3}((g \circ h)(u)) &= t_{A_3}(g(h(u))) \\
&= k_3 t_{A_3}(h(u)) \\
&= k_3 k_1 t_{A_1}(u) \\
f_{A_3}(f(u)) = f_{A_3}((g \circ h)(u)) &= f_{A_3}(g(h(u))) \\
&= k_3 f_{A_3}(h(u)) \\
&= k_3 k_1 f_{A_1}(u) \\
t_{B_3}(f(u)f(v)) = t_{B_3}((g \circ h)(u)(g \circ h)(v)) &= t_{B_3}(g(h(u))g(h(v))) \\
&= k_4 t_{B_2}(h(u)h(v)) \\
&= k_4 k_2 t_{B_1}(uv) \\
f_{B_3}(f(u)f(v)) = f_{B_3}((g \circ h)(u)(g \circ h)(v)) &= f_{B_3}(g(h(u))g(h(v))) \\
&= k_4 f_{B_2}(h(u)h(v)) \\
&= k_4 k_2 f_{B_1}(uv).
\end{aligned}$$

Thus there exists (k_3k_1, k_4k_2) morphism f from G_1 to G_3. Therefore $G_1 \approx G_2$ and hence $\approx$ is transitive. So, the relation h-morphism is an equivalence relation in the collection of vague graphs. □

Theorem 10 *Let G_1 and G_2 be two vague graphs such that G_1 is (k_1, k_2) vague morphism to G_2 for some non-zero k_1 and k_2. The image of strong edge in G_1 is strong edge in G_2 if and only if $k_1 = k_2$.*

Proof Let (u, v) be strong edge in G_1 such that $h(u)$, $h(v)$ is also strong edge in G_2. Now as $G_1 \approx G_2$ we have

$$\begin{aligned}
K_2 t_{B_1}(uv) = t_{B_2}(h(u)h(v)) &= t_{A_2}(h(u) \wedge h(v)) \\
&= k_1 t_{A_1}(u) \wedge k_1 t_{A_1}(v) \\
&= k_1 t_{B_1}(uv), \quad \text{for all } \ u \in V_1.
\end{aligned}$$

Hence, $k_2 t_{B_1}(uv) = k_1 t_{B_1}(uv)$, for all $u \in V_1$.

$$\begin{aligned} K_2 f_{B_1}(uv) = f_{B_2}((h(u)h(v))) &= f_{A_2}(h(u) \vee h(v)) \\ &= k_1 f_{A_1}(u) \vee k_1 f_{A_1}(v) \\ &= k_1 f_{B_1}(uv), \quad \text{for all} \quad u \in V_1. \end{aligned}$$

Hence $k_2 f_{B_1}(uv) = k_1 f_{B_1}(uv)$, for all $u \in V_1$.

The equations holds if and only if $k_1 = k_2$. □

Corollary 11 *Let G_1 and G_2 be two vague graphs. Let G_1 be (k_1, k_2) vague morphism to G_2. Let G_1 be strong. Then G_2 is strong if and only if $k_1 = k_2$.*

Theorem 12 *If a vague graphs G_1 is co weak isomorphic to G_2 and if G_1 is regular then G_2 is regular also.*

Proof As vague graph G_1 is co weak isomorphic to G_2, there exists a co weak isomorphism $h : G_1 \to G_2$ which is bijective that satisfies $t_{A_1}(u) \leq t_{A_2}(h(u))$ and $f_{A_1}(u) \geq f_{A_2}(h(u))$, $t_{B_1}(uv) = t_{B_2}(h(u)h(v))$ and $f_{B_1}(uv) = f_{B_2}(h(u)h(v))$, for all $u, v \in V_1$. As G_1 is regular, for $u \in V$, $\sum_{u \neq v, v \in V_1} t_{B_1}(uv) = constant$ and $\sum_{u \neq v, v \in V_1} f_{B_1}(uv) = constant$.

Now, $\sum_{h(u) \neq h(v)} t_{B_2}(h(u)h(v)) = \sum_{u \neq v, v \in V_1} t_{B_1}(uv) = constant$ and $\sum_{h(u) \neq h(v)} f_{B_2}(h(u)h(v)) = \sum_{u \neq v, v \in V_1} f_{B_1}(uv) = constant$. Therefore G_2 is regular. □

Corollary 13 *If a vague graph G_1 is co weak isomorphic to G_2 and if G_1 is strong, then G_2 need not be strong.*

Theorem 14 *Let G_1 and G_2 be two vague graphs. If G_1 is weak isomorphic to G_2 and if G_1 is strong then G_2 is strong also.*

Proof As vague graph G_1 is weak isomorphic with G_2, there exists a weak isomorphism $h : G_1 \to G_2$ which is bijective that satisfies $t_{A_1}(u) = t_{A_2}(h(u))$, $f_{A_1}(u) = f_{A_2}(h(u))$, $t_{B_1}(uv) \leq t_{B_2}(h(u)h(v))$ and $f_{B_1}(uv) \geq f_{B_2}(h(u)h(v))$. As G_1 is strong, $t_{B_1}(uv) = \min(t_{A_1}(u), t_{A_1}(v))$ and $f_{B_1}(uv) = \max(f_{A_1}(u) f_{A_1}(v))$. Now we have

$$\begin{aligned} t_{B_2}(h(u)h(v)) \geq t_{B_1}(uv) &= \min(t_{A_1}(u), t_{A_1}(v)) \\ &= \min(t_{A_2}(h(u)) t_{A_2}(h(v))). \end{aligned}$$

By the definition, $t_{B_2}(h(u)h(v)) \leq \min(t_{A_2}(h(u)), t_{A_2}(h(v)))$. Therefore, $t_{B_2}(h(u)h(v)) = \min(t_{A_2}(h(u)), t_{A_2}(h(v)))$. Similarly,

$$\begin{aligned} f_{B_2}(h(u)h(v)) \leq f_{B_1}(uv) &= \max(f_{A_1}(u), f_{A_1}(v)) \\ &= \max(f_{A_2}(h(u)) f_{A_2}(h(v))). \end{aligned}$$

By the definition, $\max(f_{A_2}(h(u)), f_{A_2}(h(v))) \leq f_{B_2}(h(u)h(v))$. Therefore, $f_{B_2}(h(u)h(v)) = \max(f_{A_2}(h(u)), f_{A_2}(h(v)))$. So, G_2 is strong. □

Corollary 15 *Let G_1 and G_2 be two vague graphs. If G_1 is weak isomorphic to G_2 and if G_1 is regular, then G_2 need not be regular.*

Theorem 16 *If the vague graph G_1 is co weak isomorphic with a strong regular vague graph G_2, then G_1 is strong regular vague graph.*

Proof As vague graph G_1 is co weak isomorphic with a vague graph G_2, there exists a co weak isomorphism $h : G_1 \to G_2$ which is bijective that satisfies $t_{A_1}(u) \le t_{A_2}(h(u))$, $f_{A_1}(u) \ge f_{A_2}(h(u))$, $t_{B_1}(uv) = t_{B_2}(h(u)h(v))$ and $f_{B_1}(uv) = f_{B_2}(h(u)h(v))$, for all $u, v \in V_1$. Now we have

$$\begin{aligned} t_{B_1}(uv) = t_{B_2}(h(u)h(v)) &= \min(t_{A_2}(h(u)), t_{A_2}(h(v))) \\ &\ge \min(t_{A_1}(u), t_{A_1}(v)) \\ f_{B_1}(uv) = f_{B_2}(h(u)h(v)) &= \max(f_{A_2}(h(u)), f_{A_2}(h(v))) \\ &\le \max(f_{A_1}(u), f_{A_1}(v)). \end{aligned}$$

But by definition $t_{B_1}(uv) \le \min(t_{A_1}(u), t_{A_1}(v))$ and $f_{B_1}(uv) \ge \max(f_{A_1}(u), f_{A_1}(v))$. So, $t_{B_1}(uv) = \min(t_{A_1}(u), t_{A_1}(v))$ and $f_{B_1}(uv) = \max(f_{A_1}(u), f_{A_1}(v))$.

Therefore, G_1 is strong. Also for $u \in V_1$, $\sum_{u \ne v, v \in V_1} t_{B_1}(uv) = \sum t_{B_2}(h(u)h(v)) =$ *constant* as G_2 is regular and $\sum_{u \ne v} f_{B_1}(uv) = \sum f_{B_2}(h(u)h(v)) =$ *constant* as G_2 is regular. Therefore G_1 s regular. □

Theorem 17 *Let G_1 and G_2 be two isomorphic vague graphs, then G_1 is strong regular if and only if G_2 is strong regular.*

Proof As a vague graph G_1 s isomorphic with vague graph G_2, there exists an isomorphism $h : G_1 \to G_2$ which is bijective and satisfies $t_{A_1}(u) = t_{A_2}(h(u))$ and $f_{A_1}(u) = f_{A_2}(h(u))$, for all $u \in V_1$ and $t_{B_1}(uv) = t_{B_2}(h(u)h(v))$ and $f_{B_1}(uv) = f_{B_2}(h(u)h(v))$, for all $uv \in E_1$. Now, G_1 is strong if and only if $t_{B_1}(uv) = \min(t_{A_1}(u), t_{A_1}(v))$ and $f_{B_1}(uv) = \max(f_{A_1}(u), f_{A_1}(v))$ if and only if $t_{B_2}(h(u)h(v)) = \min(t_{A_2}(h(u)), t_{A_2}(h(v)))$ and $f_{B_2}(h(u)h(v)) = \max(f_{A_2}(h(u)), f_{A_2}(h(v)))$ if and only if G_2 is strong. G_1 is regular if and only if $\sum_{u \ne v, v \in V_1} t_{B_1}(uv) = constant$, for all $u \in V_1$ and $\sum_{u \ne v, v \in V_1} f_{B_1}(uv) = constant$, for all $u \in V_1$ if and only if $\sum_{h(u) \ne h(v), h(v) \in V_2} t_{B_2}(h(u)h(v)) = constant$ and $\sum_{h(u) \ne h(v), h(v) \in V_2} f_{B_2}(h(u)h(v)) = constant$, for all $h(u) \in V_2$ if and only if G_2 is regular. □

Theorem 18 *A vague graphs G_1 is strong regular if and only if its complement vague graph $\overline{G}$ is strong regular vague graph also.*

Proof The complement of a vague graph is defines as $t_{A_1} = \overline{t_{A_1}}$, $f_{A_1} = \overline{f_{A_1}}$, $\overline{t_B}(uv) = t_A(u) \wedge t_A(v) - t_B(uv)$ and $\overline{f_B}(uv) = f_B(uv) - f_A(u) \vee f_A(v)$. G is strong regular if and only if $t_B(uv) = \min(t_A(u), t_A(v))$ and $f_B(uv) = \max(f_A(u), f_A(v))$ if and only if $\overline{t_B}(uv) = t_A(u) \wedge t_A(v) - t_B(uv) = t_B(uv) - t_B(uv) = 0$ and $\overline{f_B}(uv) = f_B(uv) - f_A(u) \vee f_A(v) = f_B(uv) - f_B(uv) = 0$ if and only if $\sum \overline{t_B}(uv) = 0$ and $\sum \overline{f_B}(uv) = 0$ if and only if $\overline{G}$ is strong regular vague graph. □

Definition 19 Let $G = (A, B)$ be a connected vague graph. G is said to be a highly irregular vague graph if every vertex of G is adjacent to vertices with distinct degrees.

Example 20 Consider a vague graph G such that $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. By routine computations, we have $deg(v_1) = (0.3, 1.5)$,

$deg(v_2) = (0.2, 1.3)$, $deg(v_3) = (0.4, 1.2)$, $deg(v_4) = (0.5, 1.4)$. We see that every vertex of G is adjacent to vertices with distinct degrees. So, G is highly irregular vague graph (see Fig. 3).

Theorem 21 *For any two isomorphic highly irregular vague graphs, their order and size are same.*

Proof If h from G_1 to G_2 be an isomorphism between the highly irregular vague graphs G_1 and G_2 with the underlying sets V_1 and V_2 respectively then,

$$t_{A_1}(u) = t_{A_2}(h(u)), f_{A_1}(u) = f_{A_2}(h(u)), \quad \text{for all } u \in V,$$
$$t_{B_1}(uv) = t_{B_2}(h(u)h(v)), f_{B_1}(uv) = f_{B_2}(h(u)h(v)), \quad \text{for all } u, v \in V.$$

So, we have

$$\begin{aligned}
O(G_1) &= \left(\sum_{u_1 \in V_1} t_{A_1}(u_1), \sum_{u_1 \in V_1} f_{A_1}(u_1)\right) = \left(\sum_{u_1 \in V_1} t_{A_2}(h(u_1)), \sum_{u_1 \in V_1} f_{A_2}(h(u_1))\right) \\
&= \left(\sum_{u_2 \in V_2} t_{A_2}(u_2), \sum_{u_2 \in V_2} f_{A_2}(u_2)\right) = O(G_2) \\
S(G_1) &= \left(\sum_{u_1 v_1 \in E_1} t_{B_1}(u_1 v_1), \sum_{u_1 v_1 \in E_1} f_{B_1}(u_1 v_1)\right) \\
&= \left(\sum_{u_1, v_1 \in V_1} t_{B_2}(h(u_1)h(v_1)), \sum_{u_1, v_1 \in V_1} f_{B_2}(h(u_1)h(v_1))\right) \\
&= \left(\sum_{u_2 v_2 \in E_2} t_{B_2}(u_2 v_2), \sum_{u_2 v_2 \in E_2} f_{B_2}(u_2 v_2)\right) = S(G_2).
\end{aligned}$$

□

Theorem 22 *If G_1 and G_2 are isomorphic highly irregular vague graphs then, the degrees of the corresponding vertices u and $h(u)$ are preserved.*

Proof If $h : G_1 \to G_2$ is an isomorphism between the highly irregular vague graphs G_1 and G_2 with the underlying sets V_1 and V_2 respectively then, $t_{B_1}(u_1 v_1) = t_{B_2}(h(u_1)h(v_1))$ and $f_{B_1}(u_1 v_1) = f_{B_2}(h(u_1)h(v_1))$ for all $u_1, v_1 \in V_1$. Therefore,

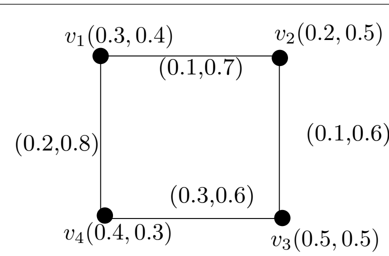

Fig. 3 A highly irregular vague graph

$$d_t(u_1) = \sum_{u_1, v_1 \in V_1} t_{B_1}(u_1v_1) = \sum_{u_1, v_1 \in V_1} t_{B_2}(h(u_1)h(v_1)) = d_t(h(u_1))$$
$$d_f(u_1) = \sum_{u_1, v_1 \in V_1} f_{B_1}(u_1v_1) = \sum_{u_1, v_1 \in V_1} f_{B_2}(h(u_1)h(v_1)) = d_f(h(u_1))$$

for all $u_1 \in V_1$. That is, the degrees of the corresponding vertices of G_1 and G_2 are the same. □

Definition 23 A vague graph G is said to be

1. self complementary if $G \cong \overline{G}$,
2. self weak complementary if G is weak isomorphic with $\overline{G}$.

Example 24 Let us consider, a vague graph $G = (A, B)$ where, the vertex set be $V = \{v_1, v_2, v_2\}$ and edge set is $\{v_1v_2, v_2v_3\}$. Obviously, the graph is self complementary (see Fig. 4). If identity bijective mapping is assumed, then G and $\overline{G}$ are weak isomorphism.

Theorem 25 *Let G be a self weak complementary highly irregular vague graph then,*

$$\sum_{u \neq v} t_B(uv) \leq \frac{1}{2} \sum_{u \neq v} \min(t_A(u), t_A(v)),$$
$$\sum_{u \neq v} f_B(uv) \geq \frac{1}{2} \sum_{u \neq v} \max(f_A(u), f_A(v)).$$

Proof Let $G = (A, B)$ be a self weak complementary highly irregular vague graph of $G^* = (V, E)$. Then, there exists a weak isomorphism $h : G \to \overline{G}$ such that for all $u, v \in V$ we have $t_A(u) = \overline{t_A}(h(u)) = t_A(h(u))$, $f_A(u) = \overline{f_A}(h(u)) = f_A(h(u)) t_B(uv) \leq \overline{t_B}(h(u)h(v))$, $f_B(uv) \geq \overline{f_B}(h(u)h(v))$.

Using the definition of complement in the above inequality, for all $u, v \in V$ we have

$$t_B(uv) \leq \overline{t_B}(h(u)h(v)) = \min(t_A(h(u)), t_A h(v)) - t_B(h(u)h(v))$$
$$f_B(uv) \geq \overline{f_B}(h(u)h(v)) = f_B(h(u)h(v)) - \max(f_A(h(u)), f_A(h(v))$$
$$t_B(uv) + t_B(h(u)h(v)) \leq \min(t_A(h(u)), t_A(h(v)))$$
$$f_B(uv) + f_B(h(u)h(v)) \geq \max(f_A(h(u)), f_A(h(v))).$$

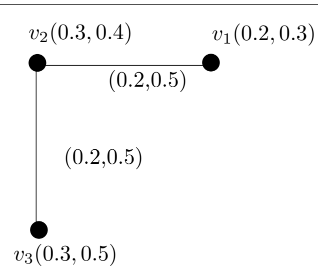

Fig. 4 An example of vague graph with self complementary and weak complementary

So, $\sum_{u\neq v} t_B(uv) + \sum_{u\neq v} t_B(h(u)h(v)) \leq \sum_{u\neq v} \min(t_A(h(u)), t_A(h(v)))$ and $\sum_{u\neq v} f_B(uv) + \sum_{u\neq v} f_B(h(u)h(v)) \geq \sum_{u\neq v} \max(f_A(h(u)), f_A(h(v)))$. Hence, $2\sum_{u\neq v} t_B(uv) \leq \sum_{u\neq v} \min(t_A(u), t_A(v))$ and $2\sum_{u\neq v} f_B(uv) \geq \sum_{u\neq v} \max(f_A(u), f_A(v))$. Now we have $\sum_{u\neq v} t_B(uv) \leq \frac{1}{2}\sum_{u\neq v} \min(t_A(u), t_A(v))$ and $\sum_{u\neq v} f_B(uv) \geq \frac{1}{2}\sum_{u\neq v} \max(f_A(u), f_A(v))$.

Definition 26 Let $G = (A, B)$ be a vague graph. The μ-complement of G is defined as $G^\mu = (A, B^\mu)$ where $B^\mu = (t_B^\mu, f_B^\mu)$ and

$$t_B^\mu(uv) = \begin{cases} t_A(u) \wedge t_A(v) - t_B(uv) & \text{if } t_B(uv) > 0 \\ 0 & \text{if } t_B(uv) = 0 \end{cases}$$

$$f_B^\mu(uv) = \begin{cases} f_B(uv) - f_A(u) \vee f_A(v) & \text{if } f_B(uv) > 0 \\ 0 & \text{if } f_B(uv) = 0 \end{cases}$$

Example 27 Let us consider a vague graph $G = (A, B)$ where the vertex set is $V = \{v_1, v_2, v_3\}$ and edge set is $E = \{v_1v_2, v_2v_3, v_1v_3\}$ (see Fig. 5).

Theorem 28 *The μ-complement of a highly irregular vague graph need not be highly irregular.*

Proof To every vertex, the adjacent vertices with distinct degrees or the non-adjacent vertices with distinct degrees may happen to be adjacent vertices with same degrees. This contradicts the definition of highly irregular vague graph. □

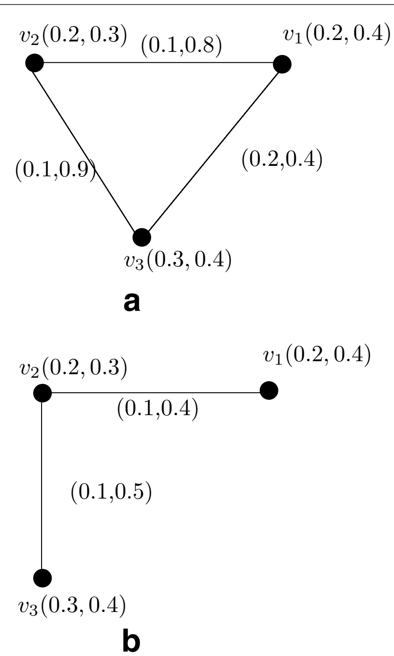

Fig. 5 μ-Complement of vague graphs. **a** Vague graph *G*. **b** μ-Complement of *G*

Theorem 29 *Let G_1 and G_2 be two highly irregular vague graphs. If G_1 and G_2 are isomorphic, then μ-complement of G_1 and G_2 are isomorphic also and vice versa.*

Proof Assume that G_1 and G_2 are isomorphic, there exists a bijective map $h : V_1 \to V_2$ satisfying $t_{A_1}(u) = t_{A_2}(h(u))$, $f_{A_1}(u) = f_{A_2}(h(u))$, for all $u \in V_1$ and $t_{B_1}(uv) = t_{B_2}(h(u)h(v))$, $f_{B_1}(uv) = f_{B_2}(h(u)h(v))$, for all $uv \in E_1$. By the definition of μ-complement we have $t^{\mu}_{B_1}(uv) = \min(t_{A_1}(u), t_{A_1}(v)) - t_{B_1}(uv) = \min(t_{A_2}(h(u)), t_{A_2}(h(v))) -t_{B_2}(h(u)h(v))$, $f^{\mu}_{B_1}(uv) = f_{B_1}(uv) - \max(f_{A_1}(u), f_{A_1}(v)) = f_{B_2}(h(u)h(v)) - \max(f_{A_2}(h(u)), f_{A_2}(h(v)))$, for all $uv \in E_1$. Hence, $G_1^{\mu} \cong G_2^{\mu}$. The proof of the converse part is straight forward. $\square$

Conclusion

It is well known that graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in computer science, physical, biological and social systems. In this paper, we introduced the notion of vague h-morphism on vague graphs and studied the action of vague h-morphism on vague strong regular graphs. We defined μ-complement of highly irregular vague graphs and investigated its properties.

Authors' contributions

The authors contributed equally to each parts of the paper. All authors read and approved the final manuscript.

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Competing interests

The authors claim that they have no competing interests.

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