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Congruences for central factorial numbers modulo powers of prime

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Abstract

Central factorial numbers are more closely related to the Stirling numbers than the other well-known special numbers, and they play a major role in a variety of branches of mathematics. In the present paper we prove some interesting congruences for central factorial numbers.

Keywords: Central factorial numbers of the first kind, Central factorial numbers of the second kind, Congruence, Stirling numbers

Mathematics Subject Classification: 11B68, 11B73, 05A10, 11B83

Introduction and definitions

Central factorial numbers are more closely related to the Stirling numbers than the other well-known special numbers, such as Bernoulli numbers, Euler numbers, trigonometric functions and their inverses. Properties of these numbers have been studied in different perspectives (see Butzer et al. 1989; Comtet 1974; Liu 2011; Merca 2012; Riordan 1968). Central factorial numbers play a major role in a variety of branches of mathematics (see Butzer et al. 1989; Chang and Ha 2009; Vogt 1989): to finite difference calculus, to approximation theory, to numerical analysis, to interpolation theory, in particular to Voronovskaja and Komleva-type expansions of trigonometric convolution integrals.

The central factorial numbers $t(n,k)$ $(k \in \mathbb{Z})$ of the first kind and $T(n,k)$ $(k \in \mathbb{Z})$ of the second kind are given by the following expansion formulas (see Butzer et al. 1989; Liu 2011; Riordan 1968)

$$x^{[n]} = \sum_{k=0}^{n} t(n,k)x^k \tag{1}$$

and

$$x^n = \sum_{k=0}^{n} T(n,k)x^{[k]}, \tag{2}$$

respectively, where $x^{[n]} = x(x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdots(x+\frac{n}{2}-n+1)$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}$ being the set of positive integers, $\mathbb{Z}$ being the set of integers.

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It follows from (1) that

$$t(n,k) = t(n-2,k-2) - \frac{1}{4}(n-2)^2 t(n-2,k) \tag{3}$$

with

$$(x^2-1^2)(x^2-2^2)\cdots(x^2-(n-1)^2) = \sum_{k=1}^{n} t(2n,2k)x^{2k-2}. \tag{4}$$

Similarly, (2) gives

$$T(n,k) = T(n-2,k-2) + \frac{1}{4}k^2 T(n-2,k) \tag{5}$$

with

$$\frac{x^{2k}}{(1-x^2)\big(1-(2x)^2\big)\cdots\big(1-(kx)^2\big)} = \sum_{n=0}^{\infty} T(2n,2k)x^{2n} \tag{6}$$

and

$$k!T(n,k) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\frac{k}{2}-i\right)^n. \tag{7}$$

Several papers obtain useful results on congruences of Stirling numbers, Bernoulli numbers and Euler numbers (see Chan and Manna 2010; Lengyel 2009; Sun 2005; Zhao et al. 2014). But only a few of congruences on central factorial numbers for odd prime moduli which can be found in (Riordan 1968, p. 236). For example, let $t_n(x) = \sum_{k=0}^{n} t(n,k)x^k$, then

$$t_p(x) \equiv x^p - x \pmod{p}, \tag{8}$$

$$t_{p+k}(x) \equiv t_p(x)\cdot t_k(x) \pmod{p}. \tag{9}$$

Conclusions

In the present paper we prove some other interesting congruences for central factorial numbers. In "Congruences for $T(ap^{m-1}(p-1)+r,k)$ modulo powers of prime p" section, some congruence relations for $T(ap^{m-1}(p-1)+r,k)$ modulo powers of prime p are established. For a is odd, $m,k \in \mathbb{N}$ and $k \le 2^{m-1}a$, we prove that

$$k!T(2^{m-1}a,k) \equiv \begin{cases} -2^{k-1} \pmod{2^m}, & k \equiv 0 \pmod{4}, \\ 2^{k-1} \pmod{2^m}, & k \equiv 2 \pmod{4}. \end{cases}$$

For p is odd prime, $m,a,k \in \mathbb{N}, r \in \mathbb{N}_0, k \le p-1$ and $r < p^{m-1}(p-1)$, in "Congruences for $T(ap^{m-1}(p-1)+r,k)$ modulo powers of prime p" section we also show that

$$T(ap^{m-1}(p-1)+r,k) \equiv T(r,k) \pmod{p^m}, \quad 1 \le r < p^{m-1}(p-1)$$

and

$$k!T(ap^{m-1}(p-1),k) \equiv (-1)^{\frac{k}{2}+1}\binom{k}{\frac{k}{2}} \pmod{p^m}, \quad \text{k is even.}$$

In "Congruences for $t(2ap^m, 2k)$ and $T(2n, 2ap^m)$ modulo powers of p" section, congruences on $t(2ap^m, 2k)$ and $T(2n, 2ap^m)$ modulo powers of p are derived. Moreover, the following results are obtained: (1) for $a, k, m \in \mathbb{N}, b \in \mathbb{N}_0$ and $2^{m-1}a \leq k \leq 2^m a$, we prove a congruence for $t(2^{m+1}a + 2b, 2k) \pmod{2^m}$; (2) for $a, n, m \in \mathbb{N}, b \in \mathbb{N}_0$ and $n \geq 2^m a$, we prove a congruence for $T(2n, 2^{m+1}a + 2b) \pmod{2^m}$; (3) for p is a odd prime number and $a, k, m \in \mathbb{N}, b \in \mathbb{N}_0$, we deduce a congruence for $t(2ap^m + 2b, 2k) \pmod{p^m}$; (4) for p is a odd prime number, $a, n, m \in \mathbb{N}, b \in \mathbb{N}_0$, we deduce a congruence for $T(2n, 2ap^m + 2b) \pmod{p^m}$.

Congruences for $T(ap^{m-1}(p-1)+r,k)$ modulo powers of prime p

Theorem 1 *For a is odd, $m, k \in \mathbb{N}$ and $k \leq 2^{m-1}a$, we have*

$$k!T(2^{m-1}a,k) \equiv \begin{cases} -2^{k-1} \pmod{2^m}, & k \equiv 0 \pmod{4}, \\ 2^{k-1} \pmod{2^m}, & k \equiv 2 \pmod{4}. \end{cases} \tag{10}$$

Proof Using Euler's Theorem, $\varphi(2^m) = 2^{m-1}$. Therefore, by Fermat's Little Theorem, we get $c^{\varphi(2^m)} = c^{2^{m-1}} \equiv 1 \pmod{2^m}$ if c is odd. Observe that, when c is even, $c^{2^{m-1}} \equiv 0 \pmod{2^m}$.

Then by (7), if $k \equiv 0 \pmod{4}$, we yield

$$\begin{aligned} k!T(2^{m-1}a,k) &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\frac{k}{2}-i\right)^{2^{m-1}a} \\ &\equiv \sum_{i=1,\ i \text{ odd}}^{k}(-1)^i\binom{k}{i} \\ &= -2^{k-1} \pmod{2^m}. \end{aligned}$$

If $k \equiv 2 \pmod{4}$, we have

$$\begin{aligned} k!T(2^{m-1}a,k) &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\frac{k}{2}-i\right)^{2^{m-1}a} \\ &\equiv \sum_{i=0,\ i \text{ even}}^{k}(-1)^i\binom{k}{i} \\ &= 2^{k-1} \pmod{2^m}. \end{aligned}$$

This completes the proof of Theorem 1.

Remark By Theorem 1 and (5), we readily get

$$k!T(2^{m-1}a+2,k) \equiv \begin{cases} -k\cdot 2^{k-3} \pmod{2^m}, & k\equiv 0 \pmod 4,\\ k\cdot 2^{k-3} \pmod{2^m}, & k\equiv 2 \pmod 4. \end{cases} \tag{11}$$

Theorem 2 *For p is odd prime, $m,a,k\in\mathbb{N}$, $r\in\mathbb{N}_0$, $k\le p-1$ and $r<p^{m-1}(p-1)$, we have*

$$T\left(ap^{m-1}(p-1)+r,k\right) \equiv T(r,k) \pmod{p^m}, \quad 1\le r<p^{m-1}(p-1), \tag{12}$$

$$k!T\left(ap^{m-1}(p-1),k\right) \equiv (-1)^{\frac{k}{2}+1}\binom{k}{\frac{k}{2}} \pmod{p^m}, \quad \text{k is even}. \tag{13}$$

Proof By Euler's Theorem and Fermat's Little Theorem, we get $a^{\varphi(p^m)} = a^{p^{m-1}(p-1)} \equiv 1 \pmod{p^m}$ if $(a,p)=1$, where (a,p) is the greatest common factor of a and p. Then by (7) and noting that $(k-2i,p)=1$, we get

$$\begin{aligned} k!T\left(ap^{m-1}(p-1)+r,k\right) &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\frac{k}{2}-i\right)^{ap^{m-1}(p-1)+r}\\ &\equiv \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\frac{k}{2}-i\right)^{r}\\ &= k!T(r,k) \pmod{p^m}. \end{aligned}$$

Observe that $(k!,p)=1$. Hence,

$$T\left(ap^{m-1}(p-1)+r,k\right) \equiv T(r,k) \pmod{p^m}.$$

The proof of (12) is complete. If $r=0$, then k is even. Therefore,

$$\begin{aligned} k!T\left(ap^{m-1}(p-1),k\right) &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\frac{k}{2}-i\right)^{ap^{m-1}(p-1)}\\ &\equiv \sum_{i=0}^{k}(-1)^i\binom{k}{i} - (-1)^{\frac{k}{2}}\binom{k}{\frac{k}{2}}\\ &= (-1)^{\frac{k}{2}+1}\binom{k}{\frac{k}{2}} \pmod{p^m}. \end{aligned}$$

The proof of (13) is complete. This completes the proof of Theorem 2.

As a direct consequence of Theorem 2, we have the following corollary.

Corollary 3 For p is odd prime, $a,k\in\mathbb{N}$ and $r\in\mathbb{N}_0$, we have

$$T(a(p-1)+r,p) \equiv \begin{cases} 0 \pmod p, & 3\le r\le p-2,\\ 1 \pmod p, & r=1. \end{cases} \tag{14}$$

$$T(a(p-1)+r,p-1) \equiv \begin{cases} 0 \pmod{p}, & 1 \le r \le p-1, \\ 1 \pmod{p}, & r=0. \end{cases} \tag{15}$$

$$T(p+2,k+2) \equiv T(p,k) \equiv 0 \pmod{p}, \quad 3 \le k \le p-1. \tag{16}$$

$$T(2p+2,k+2) \equiv T(2p,k) \equiv 0 \pmod{p}, \quad 4 \le k \le p-1. \tag{17}$$

Proof By setting $m=1$ in (12) and using (5), we have

$$\begin{aligned} T(a(p-1)+r,p) &\equiv T(a(p-1)+r-2,p-2) \\ &\equiv T(r-2,p-2) = 0 \pmod{p}, \quad (3 \le r \le p-2), \\ T(a(p-1)+1,p) &\equiv T(a(p-1)-1,p-2) \\ &\equiv T(p-2,p-2) = 1 \pmod{p}. \end{aligned}$$

The proof of (14) is complete. Setting $m=1$ and $k=p-1$ in (12), we can readily get

$$T(a(p-1)+r,p-1) \equiv 0 \pmod{p}.$$

Setting $m=1$ and $k=p-1$ in (13), and noting that $(-1)^j\binom{p-1}{j} \equiv 1 \pmod{p}$ $(j=0,1,2,\ldots,p-1), (p-1)! \equiv -1 \pmod{p}$, we have

$$T(a(p-1),p-1) \equiv 1 \pmod{p}.$$

The proof of (15) is complete. If $m=1$ and $a=r$ in (12), then

$$T(rp,k) \equiv T(r,k) \pmod{p}. \tag{18}$$

Taking $r=1,2$ in (18) and using (5), we immediately get (16) and (17). This completes the proof of Corollary 3.

Congruences for $t(2ap^m, 2k)$ and $T(2n, 2ap^m)$ modulo powers of p

To establish the main results in this section, we need to introduce the following lemmas.

Lemma 4 *If* $m \in \mathbb{N}$, *then*

$$\prod_{i=1}^{2^{m-1}} (1-((2i-1)x)^2) \equiv (1-x^2)^{2^{m-1}} \pmod{2^m}, \tag{19}$$

$$\prod_{i=1}^{2^{m-1}} (1-(2ix)^2) \equiv 1 \pmod{2^m}. \tag{20}$$

Proof We prove this lemma by induction on m. We see that (19) is true for $m=1$. Assume that it is true for $m=1,2,\ldots,j-1$. Then

$$\begin{aligned}
&\prod_{i=1}^{2^j}(1-((2i-1)x)^2)\\
&\quad=\prod_{i=1}^{2^{j-1}}(1-((2i-1)x)^2)(1-((2^j+2i-1)x)^2)\\
&\quad=\prod_{i=1}^{2^{j-1}}\left[\left(1-((2i-1)x)^2\right)^2-2^{j+1}x^2(2^{j-1}+2i-1)\left(1-((2i-1)x)^2\right)\right]\\
&\quad\equiv\left(\prod_{i=1}^{2^{j-1}}(1-((2i-1)x)^2)\right)^2 \pmod{2^{j+1}}.
\end{aligned}$$

For any polynomials $A(x)$, $B(x)$, we have $A(x)\equiv B(x) \pmod{2^m} \rightarrow (A(x))^2 \equiv (B(x))^2 \pmod{2^{m+1}}$, so we obtain the desired result. That is,

$$\prod_{i=1}^{2^j}(1-((2i-1)x)^2)\equiv\left(\prod_{i=1}^{2^{j-1}}(1-((2i-1)x)^2)\right)^2\equiv(1-x^2)^{2^j} \pmod{2^{j+1}}.$$

The proof of (19) is complete. Similarly, we can prove (20) as follows.

$$\begin{aligned}
\prod_{i=1}^{2^j}\left(1-(2ix)^2\right)&=\prod_{i=1}^{2^{j-1}}\left(1-(2ix)^2\right)\left(1-((2^j+2i)x)^2\right)\\
&=\prod_{i=1}^{2^{j-1}}\left[\left(1-(2ix)^2\right)^2-2^{j+1}x^2(2^{j-1}+2i)\left(1-(2ix)^2\right)\right]\\
&\equiv\left(\prod_{i=1}^{2^{j-1}}\left(1-(2ix)^2\right)\right)^2 \pmod{2^{j+1}}\\
&\equiv 1 \pmod{2^{j+1}}.
\end{aligned}$$

This completes the proof of Lemma 4.

Similarly, we can get the following results.

Lemma 5 *If $m\in\mathbb{N}$, then*

$$\prod_{i=1}^{2^{m-1}}\left(x^2-(2i-1)^2\right)\equiv(x^2-1)^{2^{m-1}} \pmod{2^m}, \tag{21}$$

$$\prod_{i=1}^{2^{m-1}}\left(x^2-(2i)^2\right)\equiv x^{2m} \pmod{2^m}. \tag{22}$$

We are now ready to state the following theorems.

Theorem 6 *Let $a, b, k, m \in \mathbb{N}$ and $2^{m-1}a \le k \le 2^m a$, then*

$$t(2^{m+1}a, 2k) \equiv (-1)^{k-2^{m-1}a}\binom{2^{m-1}a}{k-2^{m-1}a} \pmod{2^m}, \tag{23}$$

$$t(2^{m+1}a+2b, 2k) \equiv \sum_{j=1}^{k} t(2^{m+1}a, 2j)t(2b, 2k-2j) \pmod{2^m}. \tag{24}$$

Proof By (4) and Lemma 5, we find that

$$\sum_{k=1}^{2^m a} t(2^{m+1}a, 2k)x^{2k-2}(x^2-(2^m a)^2) = (x^2-1^2)\cdots(x^2-(2^m a-1)^2)(x^2-(2^m a)^2).$$

Thus

$$\begin{aligned}
\sum_{k=1}^{2^m a} t\left(2^{m+1}a, 2k\right)x^{2k} &\equiv \left(\prod_{i=1}^{2^m}(x^2-i^2)\right)^a \\
&= \left(\prod_{i=1}^{2^{m-1}}\left(x^2-(2i-1)^2\right)\prod_{i=1}^{2^{m-1}}\left(x^2-(2i)^2\right)\right)^a \\
&\equiv \left(x^2-1\right)^{2^{m-1}a} x^{2^m a} \\
&= \sum_{k=0}^{2^{m-1}a}(-1)^k\binom{2^{m-1}a}{k}x^{2^m a+2k} \\
&= \sum_{k=2^{m-1}a}^{2^m a}(-1)^k\binom{2^{m-1}a}{k-2^{m-1}a}x^{2k} \pmod{2^m}.
\end{aligned}$$

This completes the proof of (23). For (24), we can prove this as follows.

$$\begin{aligned}
&\sum_{k=1}^{2^m a+b} t\left(2^{m+1}a+2b, 2k\right)x^{2k-2} \\
&= \left(x^2-1^2\right)\cdots\left(x^2-(2^m a)^2\right)\left(x^2-(2^m a+1)^2\right)\cdots\left(x^2-(2^m a+b-1)^2\right) \\
&\equiv \left(x^2-1^2\right)\cdots\left(x^2-(2^m a)^2\right)(x^2-1)^2\cdots\left(x^2-(b-1)^2\right) \\
&\equiv \sum_{k=1}^{2^m a} t\left(2^{m+1}a, 2k\right)x^{2k}\sum_{k=1}^{b} t(2b, 2k)x^{2k-2} \\
&= \sum_{k=2}^{2^m a+b}\sum_{j=1}^{k} t\left(2^{m+1}a, 2j\right)t\left(2b, 2k-2j\right)x^{2k-2} \pmod{2^m}.
\end{aligned}$$

This completes the proof of Theorem 6.

Remark Taking $a = 1$ and $k = 2^{m-1}, 2^{m-1}+1, 2^{m-1}+2$ in (23), we readily get

$$\begin{aligned} t\left(2^{m+1}, 2^m\right) &\equiv 1 \pmod{2^m}, \\ t\left(2^{m+1}, 2^m+2\right) &\equiv 2^{m-1} \pmod{2^m}, \\ t\left(2^{m+1}, 2^m+4\right) &\equiv 3\cdot 2^{m-2} \pmod{2^m}, \quad m \geq 3. \end{aligned}$$

Theorem 7 Let $a, b, n, m \in \mathbb{N}$ and $n \geq 2^m a$, then

$$T\left(2n, 2^{m+1}a\right) \equiv \binom{n-2^{m-1}a-1}{n-2^m a} \pmod{2^m}, \tag{25}$$

$$T\left(2n, 2^{m+1}a+2b\right) \equiv \sum_{j=0}^{n} T\left(2j, 2^{m+1}a\right) T(2n-2j, 2b) \pmod{2^m}. \tag{26}$$

Proof By (6) and Lemma 4, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T\left(2n, 2^{m+1}a\right)x^{2n} &= \prod_{i=1}^{2^m a} \frac{x^2}{(1-(ix)^2)} \\ &\equiv \left(\prod_{i=1}^{2^m} \frac{x^2}{(1-(ix)^2)}\right)^a \\ &\equiv x^{2^{m+1}a}\left(\frac{1}{\prod_{i=1}^{2^{m-1}}(1-((2i-1)x)^2)\prod_{i=1}^{2^{m-1}}(1-(2i)^2)}\right)^a \\ &\equiv x^{2^{m+1}a}\frac{1}{(1-x^2)^{2^{m-1}a}} \\ &= \sum_{n=0}^{\infty}\binom{n+2^{m-1}a-1}{n}x^{2^{m+1}a+2n} \\ &= \sum_{n=2^m a}^{\infty}\binom{n-2^{m-1}a-1}{n-2^m a}x^{2n} \pmod{2^m}. \end{aligned}$$

This completes the proof of (25). For (26), we can prove this as follows.

$$\begin{aligned} \sum_{n=0}^{\infty} T\left(2n, 2^{m+1}a+2b\right)x^{2n} &= \prod_{i=1}^{2^m a+b} \frac{x^2}{(1-(ix)^2)} \\ &\equiv \prod_{i=1}^{2^m a} \frac{x^2}{(1-(ix)^2)} \prod_{i=1}^{b} \frac{x^2}{(1-(ix)^2)} \\ &= \sum_{n=0}^{\infty} T\left(2n, 2^{m+1}a\right)x^{2n} \sum_{n=0}^{\infty} T(2n, 2b)x^{2n} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} T\left(2j, 2^{m+1}a\right)T(2n-2j, 2b)x^{2n} \pmod{2^m}. \end{aligned}$$

This completes the proof of Theorem 7.

Remark Taking $a = 1$ and $n = 2^m + 1, 2^m + 2$ in (25), we readily get

$$T\left(2^{m+1} + 2, 2^{m+1}\right) \equiv 2^{m-1} \pmod{2^m},$$
$$T\left(2^{m+1} + 4, 2^{m+1}\right) \equiv 2^{m-2} \pmod{2^m}, \quad m \geq 3.$$

Lemma 8 *If p is a odd prime number and $m \in \mathbb{N}$, then*

$$\prod_{i=1}^{p^m}(1 - (ix)^2) \equiv \left(1 - x^{p-1}\right)^{2p^{m-1}} \pmod{p^m}. \tag{27}$$

Proof Apparently, by Lagrange congruence, we have

$$(1 - x)(1 - 2x)\cdots(1 - (p-1)x)(1 - px) \equiv \left(1 - x^{p-1}\right) \pmod{p}$$

and

$$(1 + x)(1 + 2x)\cdots(1 + (p-1)x)(1 + px) \equiv (1 - x^{p-1}) \pmod{p}.$$

Thus

$$(1 - x^2)\left(1 - (2x)^2\right)\cdots(1 - (px)^2) \equiv \left(1 - x^{p-1}\right)^2 \pmod{p}.$$

Hence (27) is true for the case $m = 1$.

Suppose that (27) is true for some $m \geq 1$. Then for the case $m + 1$,

$$\begin{aligned}
&\prod_{i=1}^{p^{m+1}}\left(1 - (ix)^2\right) \\
&= \prod_{i=1}^{p^m}\left(1 - (ix)^2\right)\left(1 - (p^m + ix)^2\right)\left(1 - (2p^m + ix)^2\right)\cdots\left(1 - ((p-1)p^m + ix)^2\right) \\
&= \prod_{i=1}^{p^m}\left[\left(1 - (ix)^2\right)^p - \left(1 - (ix)^2\right)^{p-1}\left(\sum_{j=1}^{p-1}(jp^m)^2 + 2jp^m ix\right)\right. \\
&\quad \left. + \text{terms involving powers of } p^{2m} \text{ and higher}\right].
\end{aligned}$$

For any prime p and polynomials $A(x), B(x)$, we have $A(x) \equiv B(x) \pmod{p^m}$. This implies that $(A(x))^p \equiv (B(x))^p \pmod{p^{m+1}}$. With $\sum_{j=1}^{p-1}(jp^m)^2 + 2jp^m ix \equiv 0 \pmod{p^{m+1}}$, we obtain the desired result. That is,

$$\prod_{i=1}^{p^{m+1}}\left(1 - (ix)^2\right) \equiv \left(\prod_{i=1}^{p^m} 1 - (ix)^2\right)^p \equiv \left(1 - x^{p-1}\right)^{2p^m} \pmod{p^{m+1}}.$$

This completes the proof of Lemma 8.

Similarly, we get the following results.

Lemma 9 *If p is a odd prime number and $m \in \mathbb{N}$, then*

$$\prod_{i=1}^{p^m}(x^2 - i^2) \equiv (x^p - x)^{2p^{m-1}} \pmod{p^m}. \tag{28}$$

Theorem 10 *Let p is a odd prime number and $a, b, k, m \in \mathbb{N}$, then*

$$t(2ap^m, 2k) \equiv (-1)^{\frac{2k-2ap^{m-1}}{p-1}} \binom{2ap^{m-1}}{\frac{2k-2ap^{m-1}}{p-1}} \pmod{p^m}, \tag{29}$$

$$t(2ap^m + 2b, 2k) \equiv \sum_{j=1}^{k} t(2ap^m, 2j)t(2b, 2k-2j) \pmod{p^m}, \tag{30}$$

where $2k \equiv 2ap^{m-1} \pmod{p-1}$.

Proof By (4) and Lemma 9, we find that

$$\sum_{k=1}^{ap^m} t(2ap^m, 2k)x^{2k-2}(x^2 - (ap^m)^2) = (x^2 - 1^2)\cdots(x^2 - (ap^m - 1)^2)(x^2 - (ap^m)^2).$$

Thus

$$\begin{aligned}\sum_{k=1}^{ap^m} t(2ap^m, 2k)x^{2k} &\equiv \left(\prod_{i=1}^{p^m}(x^2 - i^2)\right)^a \\ &\equiv (x^p - x)^{2ap^{m-1}} \\ &= \sum_{k=0}^{2ap^{m-1}} (-1)^k \binom{2ap^{m-1}}{k} x^{2ap^{m-1}+(p-1)k} \\ &= \sum_{k=ap^{m-1}}^{ap^m} (-1)^{\frac{2k-2ap^{m-1}}{p-1}} \binom{2ap^{m-1}}{\frac{2k-2ap^{m-1}}{p-1}} x^{2k} \pmod{p^m},\end{aligned}$$

where $2k \equiv 2ap^{m-1} \pmod{p-1}$. This completes the proof of (29).

By (4) we get

$$\begin{aligned}&\sum_{k=1}^{ap^m+b} t(2ap^m + 2b, 2k)x^{2k-2} \\ &= (x^2 - 1^2)\cdots(x^2 - (ap^m)^2)(x^2 - (ap^m + 1)^2)\cdots(x^2 - (ap^m + b - 1)^2) \\ &\equiv (x^2 - 1^2)\cdots(x^2 - (ap^m)^2)(x^2 - 1)^2)\cdots(x^2 - (b-1)^2) \\ &= \sum_{k=1}^{ap^m} t(2ap^m, 2k)x^{2k} \sum_{k=1}^{b} t(2b, 2k)x^{2k-2} \\ &= \sum_{k=2}^{ap^m+b} \sum_{j=1}^{k} t(2ap^m, 2j)t(2b, 2k-2j)x^{2k-2} \pmod{p^m}.\end{aligned}$$

This completes the proof of Theorem 10.

Remark Taking $a=1$ and $2k=2p^{m-1}, 2p^{m-1}+(p-1), 2p^{m-1}+2(p-1)$ in (29), we readily get

$$t(2p^m, 2p^{m-1}) \equiv 1 \pmod{p^m}, \tag{31}$$

$$t(2p^m, 2p^{m-1}+(p-1)) \equiv -2p^{m-1} \pmod{p^m}, \tag{32}$$

$$t(2p^m, 2p^{m-1}+2(p-1)) \equiv 2p^{2m-2}-p^{m-1} \pmod{p^m}. \tag{33}$$

Obviously ,

$$t(2p, 2) \equiv 1 \pmod{p}, \tag{34}$$

$$t(2p, p+1) \equiv -2 \pmod{p}. \tag{35}$$

If $u, v \in \mathbb{N}, 1 \le u < 2p^v$. By setting $m=1, a=p^v, 2k=2p^v+u(p-1)$ in (29), and noting that $\binom{p}{j} \equiv 1 \pmod{p}$ $(j=0,1,2,\ldots,p-1)$ with $\binom{ip+r}{jp+s} \equiv \binom{i}{j}\binom{r}{s} \pmod{p}$ $(i \ge j)$, we have

$$t(2p^{v+1}, 2p^v+u(p-1)) \equiv (-1)^u \binom{2p^v}{u} \equiv 0 \pmod{p}. \tag{36}$$

The following corollary is a direct consequence of Theorem 10.

Corollary 11 Let p be a odd prime and α be a positive integer. Then for any $1 \le k \le p^\alpha(p-1)$, we have

$$t(p^\alpha(p-1), 2k) \equiv \begin{cases} 1 \pmod{p}, & 2k \equiv 0 \pmod{p^{\alpha-1}(p-1)}, \\ 0 \pmod{p}, & \text{otherwise}. \end{cases} \tag{37}$$

Proof Let $m=1, 2a=p^{\alpha-1}(p-1)$ in (29) of Theorem 10. Then we have

$$\sum_{k=1}^{\frac{p^\alpha(p-1)}{2}} t(p^\alpha(p-1), 2k)x^{2k} \equiv \sum_{j=0}^{p^{\alpha-1}(p-1)} (-1)^j \binom{p^{\alpha-1}(p-1)}{j} x^{p^{\alpha-1}(p-1)+(p-1)j} \pmod{p}.$$

By the Lucas congruence, we obtain

$$\binom{p^{\alpha-1}(p-1)}{j} \equiv \begin{cases} \binom{p-1}{\frac{j}{p^{\alpha-1}}} \pmod{p}, & j \equiv 0 \pmod{p^{\alpha-1}}, \\ 0 \pmod{p}, & \text{otherwise}. \end{cases}$$

With

$$(-1)^j \binom{p-1}{j} \equiv 1 \pmod{p} \quad (j=0,1,2,\ldots,p-1),$$

we can deduce

$$\begin{aligned}\sum_{k=1}^{\frac{p^{\alpha}(p-1)}{2}} t(p^{\alpha}(p-1),2k)x^{2k} &\equiv \sum_{j=0}^{p-1}(-1)^j\binom{p^{\alpha-1}(p-1)}{p^{\alpha-1}j}x^{p^{\alpha-1}(p-1)(j+1)}\\ &\equiv \sum_{j=1}^{p}(-1)^{j-1}\binom{p-1}{j-1}x^{p^{\alpha-1}(p-1)j}\\ &\equiv \sum_{j=1}^{p}x^{p^{\alpha-1}(p-1)j} \pmod p,\end{aligned}$$

which is obviously equivalent to (37).

The following theorem includes the congruence relations for $T(2n,2ap^m)$ and $T(2n,2ap^m+2b$.

Theorem 12 *If p is a odd prime number, $a,b,n,m\in\mathbb{N}$, then*

$$T(2n,2ap^m)\equiv\binom{\frac{2n-2ap^{m-1}}{p-1}-1}{\frac{2n-2ap^m}{p-1}} \pmod{p^m} \tag{38}$$

and

$$T(2n,2ap^m+2b)\equiv\sum_{j=0}^{n}T(2j,2ap^m)T(2n-2j,2b) \pmod{p^m}, \tag{39}$$

where $2n\equiv 2ap^m \pmod{p-1}$.

Proof By (6) and Lemma 8, we have

$$\begin{aligned}\sum_{n=0}^{\infty}T(2n,2ap^m)x^{2n} &= \prod_{i=1}^{ap^m}\frac{x^2}{(1-(ix)^2)}\\ &\equiv\left(\prod_{i=1}^{p^m}\frac{x^2}{(1-(ix)^2)}\right)^a\\ &\equiv x^{2ap^m}\frac{1}{(1-x^{p-1})^{2ap^{m-1}}}\\ &=\sum_{n=0}^{\infty}\binom{n+2ap^{m-1}-1}{n}x^{2ap^m+(p-1)n}\\ &=\sum_{n=ap^m}^{\infty}\binom{\frac{2n-2ap^{m-1}}{p-1}-1}{\frac{2n-2ap^m}{p-1}}x^{2n} \pmod{p^m}.\end{aligned}$$

This completes the proof of (38). For (39), we can prove this as follows.

$$\begin{aligned}
\sum_{n=0}^{\infty} T(2n, 2ap^m + 2b)x^{2n} &= \prod_{i=1}^{ap^m+b} \frac{x^2}{(1-(ix)^2)} \\
&\equiv \prod_{i=1}^{ap^m} \frac{x^2}{(1-(ix)^2)} \prod_{i=1}^{b} \frac{x^2}{(1-(ix)^2)} \\
&= \sum_{n=0}^{\infty} T(2n, 2ap^m)x^{2n} \sum_{n=0}^{\infty} T(2n, 2b)x^{2n} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} T(2j, 2ap^m)T(2n-2j, 2m)x^{2n} \pmod{p^m}.
\end{aligned}$$

This completes the proof of Theorem 12.

Remark Taking $a = 1$ and $2n = 2p^m + (p-1), 2p^m + 2(p-1)$ in (38), we have

$$T(2p^m + (p-1), 2p^m) \equiv 2p^{m-1} \pmod{p^m}, \tag{40}$$

$$T(2p^m + 2(p-1), 2p^m) \equiv 2p^{2m-2} + p^{m-1} \pmod{p^m}. \tag{41}$$

Obviously ,

$$T(3p-1, 2p) \equiv 2 \pmod{p}, \tag{42}$$

$$T(4p-2, 2p) \equiv 3 \pmod{p}. \tag{43}$$

If $u \in \mathbb{N}_0, v \in \mathbb{N}$. By setting $m = 1, a = p^{v-1}, 2n = 2p^{u+v}$ in (38), and noting that $\binom{ip+r}{jp+s} \equiv \binom{i}{j}\binom{r}{s} \pmod{p} \quad (i \geq j)$, we have

$$\begin{aligned}
T(2p^{u+v}, 2p^v) &\equiv \binom{2p^{v-1}\sum_{i=0}^{u} p^i - 1}{2p^{v-1} - 1} \\
&= \frac{1}{\sum_{i=0}^{u} p^i} \binom{2p^{v-1}\sum_{i=0}^{u} p^i}{2p^{v-1}} \\
&\equiv \frac{1}{\sum_{i=0}^{u} p^i} \binom{2\sum_{i=0}^{u} p^i}{2} \\
&\equiv 1 \pmod{p}.
\end{aligned}$$

That is,

$$T(2p^{u+v}, 2p^v) \equiv 1 \pmod{p}. \tag{44}$$

Authors' contributions
Both authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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References

Butzer PL, Schmidt M, Stark EL, Vogt L (1989) Central factorial numbers; their main properties and some applications. Numer Funct Anal Optim 10:419–488
Chang CH, Ha CW (2009) Central factorial numbers and values of Bernoulli and Euler polynomials at rationals. Numer Funct Anal Optim 30(3–4):214–226
Comtet L (1974) Advanced combinatorics: the art of finite and infinite expansions (translated from the French by J. W. Nienhuys). Reidel, Dordrecht and Boston
Chan O-Y, Manna DV (2010) Congruences for Stirling numbers of the second kind. Contemp Math 517:97–111
Liu GD (2011) The *D* numbers and the central factorial numbers. Publ Math Debr 79(1–2):41–53
Lengyel T (2009) On the 2-adic order of Stirling numbers of the second kind and their differences. In: DMTCS Proc AK, pp 561–572
Merca Mircea (2012) A special case of the generalized Girard–Waring formula. J Integer Seq, Article 12.5.7, 15:7
Riordan J (1968) Combinatorial identities. Wiley, New York
Sun ZW (2005) On Euler numbers modulo powers of two. J Number Theory 115:371–380
Vogt L (1989) Approximation by linear combinations of positive convolution integrals. J Approx Theory 57(2):178–201
Zhao J, Hong S, Zhao W (2014) Divisibility by 2 of Stirling numbers of the second kind and their differences. J Number Theory 140:324–348