SpringerPlus
a SpringerOpen Journal

RESEARCH Open Access

Nonlinearities distribution Laplace transform-homotopy perturbation method

Uriel Filobello-Nino[1], Hector Vazquez-Leal[1*], Brahim Benhammouda[2], Luis Hernandez-Martinez[3], Claudio Hoyos-Reyes[1], Jose Antonio Agustin Perez-Sesma[1], Victor Manuel Jimenez-Fernandez[1], Domitilo Pereyra-Diaz[1], Antonio Marin-Hernandez[4], Alejandro Diaz-Sanchez[3], Jesus Huerta-Chua[5] and Juan Cervantes-Perez[1]

Abstract

This article proposes non-linearities distribution Laplace transform-homotopy perturbation method (NDLT-HPM) to find approximate solutions for linear and nonlinear differential equations with finite boundary conditions. We will see that the method is particularly relevant in case of equations with nonhomogeneous non-polynomial terms. Comparing figures between approximate and exact solutions we show the effectiveness of the proposed method.

Keywords: Homotopy perturbation method; Nonlinear differential equation; Approximate solutions; Laplace transform; Laplace transform homotopy perturbation method; Finite boundary conditions

Introduction

Laplace Transform (L.T.) (or operational calculus) has played an important role in mathematics (Murray 1988), not only for its theoretical interest, but also because such method allows to solve, in a simpler fashion, many problems in science and engineering, in comparison with other mathematical techniques (Murray 1988). In particular the L.T. is useful for solving ODES with constant coefficients, and initial conditions, but also can be used to solve some cases of differential equations with variable coefficients and partial differential equations (Murray 1988). On the other hand, applications of L.T. for nonlinear ordinary differential equations mainly focus to find approximate solutions, thus in reference (Aminikhan & Hemmatnezhad 2012) was reported a combination of Homotopy Perturbation Method (HPM) and L.T. method (LT-HPM), in order to obtain highly accurate solutions for these equations. However, just as with L.T; LT-HPM method has been used mainly to find solutions to problems with initial conditions (Aminikhan & Hemmatnezhad 2012; Aminikhah 2012), because it is directly related with them. Therefore (Filobello-Nino et al. 2013) presented successfully, the application of LT-HPM, in the search for approximate solutions for nonlinear problems with Dirichlet, mixed and Neumann boundary conditions defined on finite intervals. This paper introduces a modification of LT-HPM, the Nonlinearities Distribution Laplace Transform-Homotopy Perturbation Method (NDLT-HPM), which will show better results for the case of linear and non-linear differential equations with non polynomial nonhomogeneous terms. The case of equations with boundary conditions on infinite intervals, has been studied in some articles, and corresponds often to problems defined on semi-infinite ranges (Aminikhah 2011; Khan et al. 2011). However the methods of solving these problems, are different from those presented in this paper (Filobello-Nino et al. 2013). As it is widely known, the importance of research on nonlinear differential equations is that many phenomena, practical or theoretical, are of nonlinear nature. In recent years, several methods focused to find approximate solutions to nonlinear differential equations, as an alternative to classical methods, have been reported, such those based on: variational approaches (Assas 2007; He 2007; Kazemnia et al. 2008; Noorzad et al. 2008), tanh method (Evans & Raslan 2005), exp-function (Xu 2007; Mahmoudi et al. 2008), Adomian's Decomposition Method (ADM) (Adomian 1988; Babolian & Biazar 2002; Kooch & Abadyan 2012; Kooch &

* Correspondence: hvazquez@uv.mx
[1]Electronic Instrumentation and Atmospheric Sciences School, Universidad Veracruzana, Circuito Gonzalo Aguirre Beltrán S/N, 9100 Xalapa, Veracruz, México
Full list of author information is available at the end of the article

Abadyan 2011; Vanani et al. 2011; Chowdhury 2011; Elias et al. 2000), parameter expansion (Zhang & Xu 2007), HPM (Aminikhan & Hemmatnezhad 2012; Aminikhah 2012; Filobello-Nino et al. 2013; Aminikhah 2011; Khan et al. 2011; Marinca & Herisanu 2011; He 1998; He 1999; He 2006a; Vazquez-Leal et al. 2014; Belendez et al. 2009; He 2000; El-Shaed 2005; He 2006b; Vazquez-Leal et al. 2012a; Ganji et al. 2008; Fereidon et al. 2010; Sharma & Methi 2011; Biazar & Ghanbari 2012; Biazar & Eslami 2012; Araghi & Sotoodeh 2012; Araghi & Rezapour 2011; Bayat et al. 2014; Bayat et al. 2013; Vazquez-Leal et al. 2012b; Vazquez-Leal et al. 2012c; Filobello-Niño et al. 2012; Biazar & Aminikhan 2009; Biazar & Ghazvini 2009; Filobello-Nino et al. 2012; Khan & Qingbiao 2011; Madani et al. 2011; Ji Huan 2006; Feng et al. 2007; Mirmoradia et al. 2009; Vazquez-Leal et al. 2012; Vazquez-Leal et al 2013), Homotopy Analysis Method (HAM) (Rashidi et al. 2012a; Rashidi et al. 2012b; Patel et al. 2012; Hassana & El-Tawil 2011), and perturbation method (Filobello-Nino et al. 2013; Holmes 1995; Filobello-Niño et al. 2013b; Filobello-Nino et al. 2014) among many others. Also, a few exact solutions to nonlinear differential equations have been reported occasionally (Filobello-Niño et al. 2013a).

The case of Boundary Value Problems (BVPs) for nonlinear ODES includes, Michaelis Menten equation (Murray 2002; Filobello-Nino et al. 2014), that describes the kinetics of enzyme-catalyzed reactions, Gelfand's differential equation (Filobello-Nino et al. 2013; Filobello-Niño et al. 2013b) which governing combustible gas dynamics, Troesch's equation (Elias et al. 2000; Feng et al. 2007; Mirmoradia et al. 2009; Vazquez-Leal et al. 2012; Hassana & El-Tawil 2011; Erdogan & Ozis 2011), arising in the investigation of confinement of a plasma column by a radiation pressure, among many others.

In the same way, the theory of BVPs for linear ODES, is a well established branch of mathematics, with many applications. Between problems of interest, related to these equations, are found: The one-dimensional quantum problem, of a particle of mass m confined in a region of zero potential by an infinite potential at two points $x = a$ and $x = b$ (King et al. 2003), heat transfer equation (King et al. 2003), wave equation which describes for instance, transverse vibrations of a uniform stretched string between two fixed points, say $x = a$ and $x = b$ (Chow 1995; Zill Dennis 2012), the Laplace equation, which governs the temperature field corresponding to the steady state in a plate (Zill Dennis 2012), and so on. Generally, many problems expressed in terms of partial differential equations, give rise through method of separation of variables, to BVPs for linear ODES (Chow 1995; King et al. 2003; Zill Dennis 2012). From the above, it becomes a priority to investigate methods, to find handy analytical approximate solutions for linear and nonlinear ODES. With this end, we propose NDLT-HPM method, which as will be seen has good precision and requires a moderate computational work.

The paper is organized as follows. In Section 2, we introduce the standard HPM. Section 3, provides a basic idea of Nonlinearities Distribution Homotopy Perturbation Method (NDHPM). Section 4 introduces NDLT-HPM. Additionally Section 5 presents two cases study. Besides a discussion on the results is presented in Section 6. Finally, a brief conclusion is given in Section 7.

Standard HPM

The standard HPM was proposed by Ji Huan He, it was introduced like a powerful tool to approach various kinds of nonlinear problems. The HPM is considered as a combination of the classical perturbation technique and the homotopy (whose origin is in the topology), but not restricted to small parameters as occur with traditional perturbation methods. For example, HPM requires neither small parameter nor linearization, but only few iterations to obtain highly accurate solutions (He 1998; He 1999).

To figure out how HPM works, consider a general nonlinear equation in the form

$$A(u) - f(r) = 0,\ r \in \Omega, \tag{1}$$

with the following boundary conditions

$$B(u, \partial u/\partial n) = 0,\ r \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ a known analytical function and Γ is the domain boundary for Ω. A can be divided into two operators L and N, where L is linear and N nonlinear; so that (1) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Generally, a homotopy can be constructed as (He 1998; He 1999)

$$\begin{aligned} H(U,p) &= (1-p)[L(U) - L(u_0)] \\ &\quad + p[L(U) + N(U) - f(r)] \\ &= 0,\ p \in [0,1],\ r \in \Omega, \end{aligned} \tag{4}$$

or

$$\begin{aligned} H(U,p) &= L(U) - L(u_0) \\ &\quad + p[L(u_0) + N(U) - f(r)] \\ &= 0,\ p \in [0,1],\ r \in \Omega, \end{aligned} \tag{5}$$

where p is a homotopy parameter, whose values are within range of 0 and 1, u_0 is the first approximation for the solution of (3) that satisfies the boundary conditions.

Assuming that solution for (4) or (5) can be written as a power series of p.

$$U = v_0 + v_1 p + v_2 p^2 + \ldots \tag{6}$$

Substituting (6) into (5) and equating identical powers of p terms, there can be found values for the sequence $v_0, v_1, v_2,$

When $p \to 1$, it yields the approximate solution for (1) in the form

$$U = v_0 + v_1 + v_2 + v_3 \ldots \tag{7}$$

Basic idea of NDHPM

(Vazquez-Leal et al. 2012b) introduced a modified version of HPM, which sometimes eases the solutions searching process for (3) and reduces the complexity of solving differential equations in terms of power series.

As first step, a homotopy of the form (Vazquez-Leal et al. 2012b) is introduced

$$\begin{aligned} H(U,p) &= (1-p)[L(U) - L(u_0)] \\ &\quad + p[L(U) + N(U,p) - f(r,p)] = 0, \end{aligned} \tag{8}$$

or

$$\begin{aligned} H(U,p) &= L(U) - L(u_0) + p[L(u_0) + N(U,p) - f(r,p)] \\ &= 0,\ p \in [0,1],\ r \in \Omega. \end{aligned} \tag{9}$$

It can be noticed that the homotopy function (8) is essentially the same as (4), except for the non-linear operator N and the non homogeneous function f, which contain embedded the homotopy parameter p. The standard procedure for the HPM is used in the rest of the method.

We propose that (He 1998; Vazquez-Leal et al. 2012b)

$$U = v_0 + v_1 p + v_2 p^2 + \ldots \tag{10}$$

When $p \to 1$, it is expected to get an approximate solution for (3) in the form

$$U = v_0 + v_1 + v_2 + v_3 + \ldots \tag{11}$$

Non-linearities distribution Laplace transform-homotopy perturbation method (NDLT-HPM)

A way to introduce, NDLT-HPM is assume that NDLT-HPM follows the same steps of NDHPM until (9), next we apply Laplace transform on both sides of homotopy equation (9), to obtain

$$\mathfrak{I}\{L(U) - L(u_0) + p[L(u_0) + N(U,p) - f(r,p)\} = 0, \tag{12}$$

(more generally, one could substitute $f(r,p)$ in (8) by another function $g(r,p)$ such that, $g(r,p) \to f(r)$ when $p \to 1$, see cases study above).

Using the differential property of L.T, we have (Murray 1988)

$$\begin{aligned} &s^n \mathfrak{I}\{U\} - s^{n-1}U(0) - s^{n-2}U'(0) - \ldots - U^{(n-1)}(0) \\ &\quad = \mathfrak{I}\{L(u_0) - pL(u_0) + p[-N(U,p) + f(r,p)]\}, \end{aligned} \tag{13}$$

or

$$\mathfrak{I}(U) = \left(\frac{1}{s^n}\right)\left\{\begin{matrix} s^{n-1}U(0) + s^{n-2}U'(0) + \ldots + U^{(n-1)}(0) \\ + \mathfrak{I}\{L(u_0) - pL(u_0) + p[-N(U,p) + f(r,p)]\} \end{matrix}\right\} \tag{14}$$

applying inverse Laplace transform to both sides of (14), we obtain

$$U = \mathfrak{I}^{-1}\left\{\begin{matrix} \left(\frac{1}{s^n}\right)\{s^{n-1}U(0) + s^{n-2}U'(0) + \ldots + U^{(n-1)}(0) \\ + \mathfrak{I}\{L(u_0) - pL(u_0) + p[-N(U,p) + f(r,p)]\}\} \end{matrix}\right\} \tag{15}$$

Assuming that the solutions of (3) and $f(r,p)$ can be expressed as a power series of p

$$U = \sum_{n=0}^{\infty} p^n v_n, \tag{16}$$

$$f = \sum_{m=0}^{\infty} p^m f_m(r). \tag{17}$$

Then substituting (16) and (17) into (15), we get

$$\sum_{n=0}^{\infty} p^n v_n = \mathfrak{I}^{-1}\left\{\begin{matrix} \left(\frac{1}{s^n}\right)\{s^{n-1}U(0) + s^{n-2}U'(0) + \ldots + U^{(n-1)}(0)\} \\ + \left(\frac{1}{s^n}\right)\mathfrak{I}\left\{L(u_0) - pL(u_0)\right. \\ \left. + p\left[-N\left(\sum_{n=0}^{\infty} p^n v_n, p\right) + \sum_{m=0}^{\infty} p^m f_m(r)\right]\right\} \end{matrix}\right\}, \tag{18}$$

comparing coefficients of p, with the same power leads to

$$p^0 : v_0 = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^n}\right)(s^{n-1}U(0) + s^{n-2}U'(0) + \ldots + U^{(n-1)}(0)) + \mathfrak{I}\{L(u_0)\})\right\},$$

$$p^1 : v_1 = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^n}\right)(\mathfrak{I}\{N(v_0) - L(u_0) + f_0(r)\})\right\},$$

$$p^2 : v_2 = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^n}\right)\mathfrak{I}\{N(v_0, v_1, f_0, f_1)\}\right\},$$

$$\begin{aligned} &p^3 : v_3 = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^n}\right)\mathfrak{I}\{N(v_0, v_1, v_2, f_0, f_1, f_2)\}\right\},\\ &\vdots\\ &p^j : v_j = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^n}\right)\mathfrak{I}\left\{N\left(v_0, v_1, v_2, ..., v_j, f_0, f_1, f_2, ..f_j\right)\right\}\right\},\\ &\vdots \end{aligned} \tag{19}$$

Assuming that the initial approximation has the form: $U(0) = u_0 = \alpha_0$, $U'(0) = \alpha_1$,.., $U^{n-1}(0) = \alpha_{n-1}$; therefore the exact solution may be obtained as follows

$$u = \lim_{p\to 1} U = v_0 + v_1 + v_2 + ... \tag{20}$$

LT-HPM is derived in a similar way to NDLT-HPM, the difference is that in the first case the Laplace transform applies to (5) instead of (9). From here on, takes place in essence the same procedure followed by NDLT-HPM (12), (13), (14), (15), (16), (17), (18) and (19) (Aminikhan & Hemmatnezhad 2012; Aminikhah 2012; Filobello-Nino et al. 2013; Aminikhah 2011).

Cases study

Next, NDLT-HPM, and LT-HPM are compared with the following two cases study

CASE STUDY 1

We will find an approximate solution the following nonlinear second order ordinary differential equation

$$\frac{d^2y(x)}{dx^2} - y^2(x) - e^x = 0,\ 0 \le x \le 1,\ y(0) = 0,\ y(1) = 2. \tag{21}$$

Method 1 Employing LT-HPM

To obtain an approximate solution for (21) by applying the LTHPM method, we identify

$$L(y) = y''(x), \tag{22}$$

$$N(y) = -y^2(x) - e^x, \tag{23}$$

where prime denotes differentiation respect to x.

To solve approximately (21), first we expand the exponential term, resulting

$$y'' - y^2(x) - \left(1 + x + \frac{1}{2}x^2 + ..\right) = 0,\ 0 \le x \le 1,\ y(0) = 0,\ y(1) = 2. \tag{24}$$

We construct the following homotopy in accordance with (4)

$$(1-p)\left(y'' - y_0''\right) + p\left[y'' - y^2 - 1 - x - \frac{1}{2}x^2\right] = 0, \tag{25}$$

or

$$y'' = y_0'' + p\left[-y_0'' + y^2 + 1 + x + \frac{x^2}{2}\right], \tag{26}$$

where we have kept three terms of Taylor series.

Applying Laplace transform to (26) we get

$$\mathfrak{I}\left(y''\right) = \mathfrak{I}\left(y_0'' + p\left(-y_0'' + y^2 + 1 + x + \frac{x^2}{2}\right)\right). \tag{27}$$

As it is explained in (Murray 1988), it is possible to rewrite (27) as

$$s^2Y(s) - sy(0) - y'(0) = \mathfrak{I}\left(y_0'' + p\left(-y_0'' + y^2 + 1 + x + \frac{x^2}{2}\right)\right), \tag{28}$$

where we have defined $Y(s) = \mathfrak{I}(y(x))$.

After applying the initial condition $y(0) = 0$, the last expression can be simplified as follows

$$s^2Y(s) - A = \mathfrak{I}\left(y_0'' + p\left(-y_0'' + y^2 + 1 + x + \frac{x^2}{2}\right)\right), \tag{29}$$

where, we have defined $A = y'(0)$.

Solving for $Y(s)$ and applying Laplace inverse transform $\mathfrak{I}^{-1}$

$$y(x) = \mathfrak{I}^{-1}\left\{\frac{A}{s^2} + \frac{1}{s^2}\left(\mathfrak{I}\left(y_0'' + p\left(-y_0'' + y^2 + 1 + x + \frac{x^2}{2}\right)\right)\right)\right\}. \tag{30}$$

Next, suppose that the solution for (30) has the form

$$y(x) = \sum_{n=0}^{\infty} p^n v_n, \tag{31}$$

and choosing

$$v_0(x) = Ax, \tag{32}$$

as the first approximation for the solution of (21) that satisfies the condition $y(0) = 0$.

Substituting (31) and (32) into (30), we get

$$\sum_{n=0}^{\infty} p^n v_n = \mathfrak{I}^{-1}\left\{\frac{A}{s^2} + \frac{1}{s^2}\mathfrak{I}\left(y_0'' + p\left(-y_0'' + \left(v_0 + pv_1 + p^2v_2 + ..\right)^2 + 1 + x + \frac{x^2}{2}\right)\right)\right\}. \tag{33}$$

Equating terms with identical powers of p, we obtain

$$p^0 : v_0(x) = \mathfrak{I}^{-1}\left\{\frac{A}{s^2}\right\}, \tag{34}$$

$$p^1 : v_1(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^2}\right)\mathfrak{I}\left(v_0^2 + 1 + x + \frac{x^2}{2}\right)\right\}, \tag{35}$$

$$p^2 : \nu_2(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^2}\right)\mathfrak{I}(2\nu_0\nu_1)\right\}, \tag{36}$$

$$p^3 : \nu_3(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^2}\right)\mathfrak{I}\left(\nu_1^2 + 2\nu_0\nu_2\right)\right\}, \tag{37}$$

$$p^4 : \nu_4(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^2}\right)\mathfrak{I}(2\nu_0\nu_3 + 2\nu_1\nu_2)\right\}, \tag{38}$$

From above we solve for $\nu_0(x)$, $\nu_1(x)$, $\nu_2(x)$,.. we obtain

$$p^0 : \nu_0(x) = Ax, \tag{39}$$

$$p^1 : \nu_1(x) = \left(A^2 + \frac{1}{2}\right)\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2}, \tag{40}$$

$$p^2 : \nu_2(x) = \frac{A}{252}\left(A^2 + \frac{1}{2}\right)x^7 + \frac{A}{90}x^6 + \frac{A}{20}x^5, \tag{41}$$

$$\begin{aligned} p^3 : \nu_3(x) = {} & \left(A^2 + \frac{1}{2}\right)\left(\frac{A^2}{6048} + \frac{1}{25920}\right)x^{10} + \left(\frac{A^2}{1440} + \frac{1}{5184}\right)x^9 \\ & + \left(\frac{11A^2}{3360} + \frac{5}{4032}\right)x^8 + \frac{x^7}{252} + \frac{x^6}{120}, \end{aligned} \tag{42}$$

$$\begin{aligned} p^4 : \nu_4(x) = {} & \left[(2A^2+1)A\left(\frac{2A^2+1}{943488} + \frac{1}{156}\left(\frac{A^2}{6048} + \frac{1}{25920}\right)\right)\right]x^{13} \\ & + \left[\frac{23A^3}{665280} + 1.4947758 \times 10^{-5}A\right]x^{12} \\ & + \left[\frac{11A^3}{665280} + 1.211332371 \times 10^{-4}A\right]x^{11} \\ & + \frac{A}{2520}x^{10} + \frac{A}{1080}x^9, \\ & \vdots \end{aligned} \tag{43}$$

and so on.

By substituting solutions (39), (40), (41), (42) and (43) into (20) results in a fourth order approximation

$$\begin{aligned} y(x) = {} & Ax + \frac{x^2}{2} + \frac{x^3}{6} + \left(A^2 + \frac{1}{2}\right)\frac{x^4}{12} + \frac{A}{20}x^5 + \left(\frac{A}{90} + \frac{1}{120}\right)x^6 \\ & + \left(\frac{A}{252}\left(A^2 + \frac{1}{2}\right) + \frac{1}{252}\right)x^7 + \left(\frac{11A^2}{3360} + \frac{5}{4032}\right)x^8 \\ & + \left(\frac{A^2}{1440} + \frac{1}{5184} + \frac{A}{1080}\right)x^9 \\ & + \left(\left(A^2 + \frac{1}{2}\right)\left(\frac{A^2}{6048} + \frac{1}{25920}\right) + \frac{A}{2520}\right)x^{10} \\ & + \left[\frac{11A^3}{665280} + 1.211332371 \times 10^{-4}A\right]x^{11} \\ & + \left[\frac{23A^3}{665280} + 1.4947758 \times 10^{-5}A\right]x^{12} \\ & + \left[(2A^2+1)A\left(\frac{2A^2+1}{943488} + \frac{1}{156}\left(\frac{A^2}{6048} + \frac{1}{25920}\right)\right)\right]x^{13}. \end{aligned} \tag{44}$$

In order to calculate the value of A, we require that (44) satisfies the boundary condition $y(1) = 2$, so that we obtain

$$A = 1.096310072. \tag{45}$$

Method 2 Employing NDLT-HPM

In accordance with NDLT-HPM, we propose the following homotopy

$$(1-p)\left(y'' - y_0''\right) + p\left[y'' - y^2 - pe^{px}\right] = 0, \tag{46}$$

we see that (46) is not exactly of the form (8), but note that $g(x, p) = pe^{px} \to e^x$, if $p \to 1$.

After expanding the exponential term, we obtain

$$(1-p)\left(y'' - y_0''\right) + p\left[y'' - y^2 - p\left(1 + xp + \frac{1}{2}p^2x^2\right)\right] = 0, \tag{47}$$

or

$$y'' = y_0'' + p\left[-y_0'' + y^2 + p + p^2x + \frac{p^3x^2}{2}\right]. \tag{48}$$

Applying Laplace transform to (48), we get

$$\mathfrak{I}\left(y''\right) = \mathfrak{I}\left(y_0'' + p\left(-y_0'' + y^2 + p + p^2x + \frac{p^3x^2}{2}\right)\right), \tag{49}$$

it is possible to rewrite (49) as

$$s^2Y(s) - sy(0) - y'(0) = \mathfrak{I}\left(y_0'' + p\left(-{}^{0''}_{y} + y^2 + p + p^2x + \frac{p^3x^2}{2}\right)\right), \tag{50}$$

where we have defined $Y(s) = \mathfrak{I}(y(x))$.

Applying the initial condition $y(0) = 0$, (50) can be simplified as follows

$$s^2Y(s) - A = \mathfrak{I}\left(y_0'' + p\left(-y_0'' + y^2 + p + p^2x + \frac{p^3x^2}{2}\right)\right), \tag{51}$$

where, we have defined $A = y'(0)$.

Solving for $Y(s)$ and applying Laplace inverse transform $\mathfrak{I}^{-1}$

$$\begin{aligned} y(x) = \mathfrak{I}^{-1}\Bigg\{ & \frac{A}{s^2} + \frac{1}{s^2}\Bigg(\mathfrak{I}\Bigg(y_0'' + p\Bigg(-y_0'' + y^2 \\ & + p + p^2x + \frac{p^3x^2}{2}\Bigg)\Bigg)\Bigg)\Bigg\}. \end{aligned} \tag{52}$$

Assuming that the solution for (52) has the form

$$y(x) = \sum_{n=0}^{\infty} p^n \nu_n, \tag{53}$$

and choosing

$$\nu_0(x) = Ax, \tag{54}$$

as the first approximation for the solution of (21) that satisfies the condition $y(0) = 0$.

Substituting (53) and (54) into (52), we get

$$\sum_{n=0}^{\infty} p^n v_n = \mathfrak{I}^{-1}\left\{\frac{A}{s^2}+\frac{1}{s^2}\mathfrak{I}\left(y_0^{''}+p\left(-y_0^{''}+\left(v_0+pv_1+p^2v_2+..\right)^2+p+p^2x+\frac{p^3x^2}{2}\right)\right)\right\}. \tag{55}$$

Equating terms with identical powers of p, we obtain

$$p^0 : v_0(x) = \mathfrak{I}^{-1}\left\{\frac{A}{s^2}\right\}, \tag{56}$$

$$p^1 : v_1(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^2}\right)\mathfrak{I}\left(v_0^2\right)\right\}, \tag{57}$$

$$p^2 : v_2(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^2}\right)\mathfrak{I}\left(2v_0v_1+1\right)\right\}, \tag{58}$$

$$p^3 : v_3(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^2}\right)\mathfrak{I}\left(v_1^2+2v_0v_2+x\right)\right\}, \tag{59}$$

$$p^4 : v_4(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^2}\right)\mathfrak{I}\left(2v_0v_3+2v_1v_2+\frac{x^2}{2}\right)\right\},$$
$$\vdots \tag{60}$$

Solving the above equations for $v_0(x), v_1(x), v_2(x)$ ..., we obtain

$$p^0 : v_0(x) = Ax, \tag{61}$$

$$p^1 : v_1(x) = \frac{A^2}{12}x^4, \tag{62}$$

$$p^2 : v_2(x) = \frac{A^3}{252}x^7+\frac{1}{2}x^2, \tag{63}$$

$$p^3 : v_3(x) = \frac{135A^4}{816480}x^{10}+\frac{A}{20}x^5+\frac{1}{6}x^3, \tag{64}$$

$$p^4 : v_4(x) = \frac{A^5}{157248}x^{13}+\frac{11A^2}{3360}x^8+\frac{A}{90}x^6+\frac{x^4}{24}, \tag{65}$$

and so on.

By substituting solutions (61), (62), (63), (64) and (65) into (20) results in a fourth order approximation

$$y(x) = Ax+\frac{x^2}{2}+\frac{x^3}{6}+\frac{A^2+1/2}{12}x^4+\frac{A}{20}x^5+\frac{A}{90}x^6+\frac{A^3}{252}x^7+\frac{11A^2}{3360}x^8+\frac{A^4}{6048}x^{10}+\frac{A^5}{157248}x^{13}. \tag{66}$$

In order to calculate the value of A, we require that (66) satisfies the boundary condition $y(1) = 2$, so that we obtain

$$A = 1.111131377. \tag{67}$$

Case study 2

We will find an approximate solution for the following linear third order ordinary differential equation with variable coefficients.

$$\frac{d^3y(x)}{dx^3}-xy(x)+x^2\sin x = 0,\ 0\le x\le 1,$$
$$y(0) = 0, y'(0) = 1y(1) = 2. \tag{68}$$

Method 1 Employing LT-HPM

To obtain a solution for (68) by applying the LT-HPM method, we identify

$$L(y) = y'''(x), \tag{69}$$

$$N(y) = -xy(x)+x^2\sin x, \tag{70}$$

where prime denotes differentiation respect to x.

To solve approximately (68), first we expand the trigonometric term, resulting

$$y'''-xy(x)+x^2\left(x-\frac{1}{6}x^3+..\right) = 0,\ 0\le x\le 1,$$
$$y(0) = 0, y'(0) = 1y(1) = 2. \tag{71}$$

We construct the following homotopy in accordance with (4)

$$(1-p)\left(y'''-y_0'''\right)+p\left[y'''-xy+x^3-\frac{x^5}{6}\right] = 0, \tag{72}$$

where we have kept just two terms of Taylor series, or

$$y''' = y_0'''+p\left[-y_0'''+xy-x^3+\frac{x^5}{6}\right]. \tag{73}$$

Applying Laplace transform to (73), we get

$$\mathfrak{I}\left(y'''\right) = \mathfrak{I}\left(y_0'''+p\left(-y_0'''+xy-x^3+\frac{x^5}{6}\right)\right). \tag{74}$$

In accordance with (Murray 1988), it is possible to rewrite (74) as

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0) = \mathfrak{I}\left(y_0'''+p\left(-y_0'''+xy-x^3+\frac{x^5}{2}\right)\right). \tag{75}$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 1$, (75) adopts the following form

$$s^3Y(s)-s-A = \mathfrak{I}\left(y_0'''+p\left(-y_0'''+xy-x^3+\frac{x^5}{2}\right)\right), \tag{76}$$

where, we have defined $A = y''(0)$.

Solving for $Y(s)$ and applying Laplace inverse transform $\mathfrak{I}^{-1}$

$$y(x) = \mathfrak{I}^{-1}\left\{\frac{A}{s^3} + \frac{1}{s^2} + \frac{1}{s^3}\left(\mathfrak{I}\left(y_0''' + p\left(-y_0''' + xy - x^3 + \frac{x^5}{6}\right)\right)\right)\right\}. \tag{77}$$

Assuming that the solution for (77) has the form

$$y(x) = \sum_{n=0}^{\infty} p^n v_n, \tag{78}$$

and choosing

$$v_0(x) = \frac{A}{2}x^2 + x, \tag{79}$$

let be the first approximation for the solution of (68) that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$.

Substituting (78) and (79) into (77), we get

$$\begin{aligned}\sum_{n=0}^{\infty} p^n v_n = \mathfrak{I}^{-1}\Bigg\{ &\frac{A}{s^3} + \frac{1}{s^2} + \frac{1}{s^3}\mathfrak{I}\Bigg(y_0''' \\ &+ p\Bigg(-y_0''' + x(v_0 + pv_1 \\ &+ p^2 v_2 + \ldots) - x^3 + \frac{x^5}{6}\Bigg)\Bigg)\Bigg\}.\end{aligned} \tag{80}$$

Equating terms with identical powers of p, we obtain

$$p^0 : v_0(x) = \mathfrak{I}^{-1}\left\{\frac{1}{s^2} + \frac{A}{s^3}\right\}, \tag{81}$$

$$p^1 : v_1(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3}\right)\mathfrak{I}\left(xv_0 - x^3 + \frac{x^5}{6}\right)\right\}, \tag{82}$$

$$p^2 : v_2(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3}\right)\mathfrak{I}(xv_1)\right\}, \tag{83}$$

$$p^3 : v_3(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3}\right)\mathfrak{I}(xv_2)\right\}, \tag{84}$$

$$p^4 : v_4(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3}\right)\mathfrak{I}(xv_3)\right\}, \tag{85}$$

From above we solve for $v_0(x), v_1(x), v_2(x) \ldots$, we obtain

$$p^0 : v_0(x) = \frac{A}{2}x^2 + x, \tag{86}$$

$$p^1 : v_1(x) = \frac{A/2-1}{120}x^6 + \frac{x^5}{60} + \frac{x^8}{2016}, \tag{87}$$

$$p^2 : v_2(x) = \frac{A/2-1}{86400}x^{10} + \frac{x^9}{30240} + \frac{x^{12}}{2661120}, \tag{88}$$

$$p^3 : v_3(x) = \frac{x^{13}}{51891840} + \frac{A/2-1}{188697600}x^{14} + \frac{x^{16}}{8941363200}, \tag{89}$$

$$\begin{aligned}p^4 : v_4(x) = {} & 4.723248187 \times 10^{-12}x^{17} \\ & + 1.082411043 \times 10^{-12}(A/2-1)x^{18} \\ & + 1.635084351 \times 10^{-14}x^{20},\end{aligned} \tag{90}$$

and so on.

By substituting solutions (86), (87), (88), (89) and (90) into (20) results in a fourth order approximation

$$\begin{aligned}y(x) = {} & x + \frac{A}{2}x^2 + \frac{x^5}{60} + \frac{A/2-1}{120}x^6 + \frac{x^8}{2016} + \frac{x^9}{30240} + \frac{A/2-1}{86400}x^{10} \\ & + \frac{x^{12}}{2661120} + \frac{x^{13}}{51891840} + \frac{A/2-1}{188697600}x^{14} \\ & + \frac{x^{16}}{8941363200} + 4.723248187 \times 10^{-12}x^{17} + 1.082411043 \\ & \times 10^{-12}(A/2-1)x^{18} + 1.635084351 \times 10^{-14}x^{20}.\end{aligned} \tag{91}$$

In order to calculate the value of A, we require that (91) satisfies the boundary condition $y(1) = 2$, so that we obtain

$$A = 1.965892301. \tag{92}$$

Method 2 Employing NDLT-HPM

In accordance with NDLT-HPM, it is possible to propose the following homotopy (see (9))

$$(1-p)\left(y''' - y_0'''\right) + p\left[y''' - xy + x^2 p g(x,p)\right] = 0, \tag{93}$$

where we have defined

$$g(x,p) = p^2 \sin px + x^3\left(\frac{p^5 - p^3}{6}\right), \tag{94}$$

with the property

$$\lim_{p\to 1} g(x,p) = \sin x. \tag{95}$$

after expanding the two first terms of sin function, we obtain

$$(1-p)\left(y''' - y_0'''\right) + p\left[y''' - xy + x^2 p^3\left(x - x^3/6\right)\right] = 0, \tag{96}$$

or

$$y'' = y_0'' + p\left[-y_0'' + xy - x^3 p^3 + \frac{p^3 x^5}{6}\right]. \tag{97}$$

Applying Laplace transform to (97) we get

$$\mathfrak{I}\left(y''\right) = \mathfrak{I}\left(y_0''' + p\left(-y_0''' + xy - x^3 p^3 + \frac{p^3 x^5}{6}\right)\right), \tag{98}$$

it is possible to rewrite (98) as

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0) = \mathfrak{I}\left(y_0''' + p\left(-y_0''' + xy - x^3p^3 + \frac{p^3x^5}{6}\right)\right), \quad (99)$$

where once again, we have defined $Y(s) = \mathfrak{I}(y(x))$.

Applying the initial conditions $y(0) = 0$, and $y'(0) = 1$, (99) can be simplified as follows

$$s^3Y(s)-s-A = \mathfrak{I}\left(y_0''' + p\left(-y_0''' + xy - x^3p^3 + \frac{p^3x^5}{6}\right)\right), \quad (100)$$

where, we have defined $A = y''(0)$.

Solving for $Y(s)$ and applying Laplace inverse transform $\mathfrak{I}^{-1}$

$$y(x) = \mathfrak{I}^{-1}\left\{\frac{1}{s^2} + \frac{A}{s^3} + \frac{1}{s^3}\mathfrak{I}\left(y_0''' + p\left(-y_0''' + xy - x^3p^3 + \frac{p^3x^5}{6}\right)\right)\right\}. \quad (101)$$

Next, we assume a series solution for $y(x)$, in the form

$$y(x) = \sum_{n=0}^{\infty} p^n v_n, \quad (102)$$

let

$$v_0(x) = \frac{A}{2}x^2 + x, \quad (103)$$

be the first approximation for the solution of (68) that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$.

Substituting (102) and (103) into (101), we get

$$\sum_{n=0}^{\infty} p^n v_n = \mathfrak{I}^{-1}\left\{\frac{A}{s^3} + \frac{1}{s^2} + \frac{1}{s^3}\mathfrak{I}\left(y_0''' + p\left(-y_0''' + x\left(v_0 + pv_1 + p^2v_2 + \ldots\right) - x^3p^3 + \frac{p^3x^5}{6}\right)\right)\right\}. \quad (104)$$

On comparing the coefficients of like powers of p we have

$$p^0 : v_0(x) = \mathfrak{I}^{-1}\left\{\frac{1}{s^2} + \frac{A}{s^3}\right\}, \quad (105)$$

$$p^1 : v_1(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3}\right)\mathfrak{I}(xv_0)\right\}, \quad (106)$$

$$p^2 : v_2(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3}\right)\mathfrak{I}(xv_1)\right\}, \quad (107)$$

$$p^3 : v_3(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3}\right)\mathfrak{I}(xv_2)\right\}, \quad (108)$$

$$p^4 : v_4(x) = \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3}\right)\mathfrak{I}\left(xv_3 - x^3 + \frac{x^5}{6}\right)\right\}, \quad (109)$$

Performing the above operations for $v_0(x), v_1(x), v_2(x) \ldots$, we obtain

$$p^0 : v_0(x) = \frac{A}{2}x^2 + x, \quad (110)$$

$$p^1 : v_1(x) = \frac{x^5}{60} + \frac{A}{240}x^6, \quad (111)$$

$$p^2 : v_2(x) = \frac{x^9}{30240} + \frac{A}{172800}x^{10}, \quad (112)$$

$$p^3 : v_3(x) = \frac{x^{13}}{51891840} + \frac{A}{377395200}x^{14}, \quad (113)$$

$$p^4 : v_4(x) = -\frac{x^6}{120} + \frac{x^8}{2016} + \frac{x^{17}}{211718707200} + \frac{A}{1847726899200}x^{18}, \quad (114)$$

and so on.

By substituting solutions (110), (111), (112), (113) and (114) into (20) and calculating the limit when $p \to 1$, results in a fourth order approximation

$$y(x) = x + \frac{A}{2}x^2 + \frac{1}{60}x^5 + \left(\frac{A}{240} - \frac{1}{120}\right)x^6 + \frac{1}{2016}x^8 + \frac{1}{30240}x^9 + \frac{A}{172800}x^{10} + \frac{1}{51891840}x^{13} + \frac{Ax^{14}}{377395200} + \frac{1}{211718707200}x^{17} + \frac{Ax^{18}}{1847726899200}. \quad (115)$$

In order to calculate the value of A, we require that (115) satisfies the boundary condition $y(1) = 2$, resulting an equation for A, from which we obtain the following result

$$A = 1.965892301. \quad (116)$$

Discussion

This work showed the accuracy of NDLT-HPM in solving ordinary differential equations with nonhomogeneous non-polynomial terms and finite boundary conditions, and it can be considered as a continuation of (Filobello-Nino et al. 2013) where in principle, LT-HPM already provided the possibility of solving problems, with the nonhomogeneities mentioned in this study (Aminikhan & Hemmatnezhad 2012; Aminikhah 2012; Filobello-Nino et al. 2013; Aminikhah 2011), but were not carried out. One way to introduce LT-HPM to this kind of problems, is directly apply the Laplace transform to the homotopy equation (5) and then following a procedure identical to that applied in (Filobello-Nino et al. 2013) (see also (12),

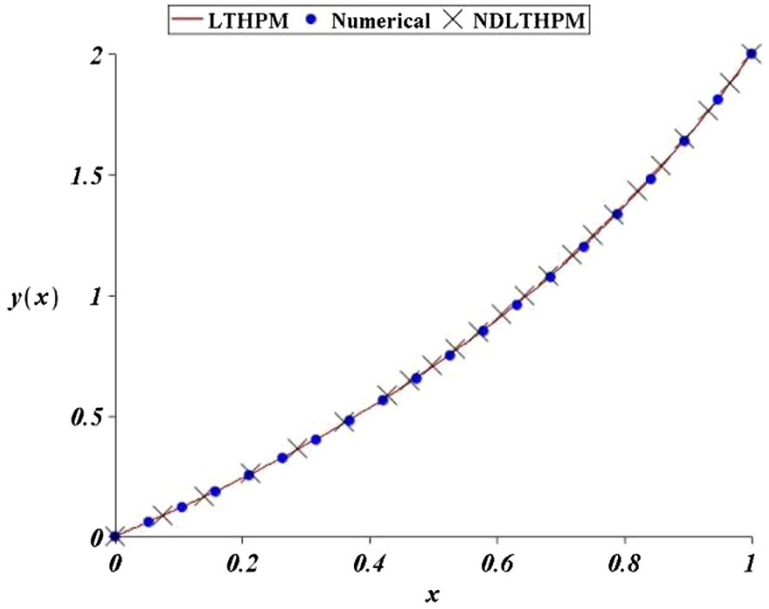

Figure 1 Comparison between numerical solution of (21) and LT-HPM, NDLT-HPM approximations (44), (66).

(13), (14), (15), (16), (17), (18) and (19)), although a possible difficulty is that, the mathematical procedure becomes long and cumbersome, depending on the function (see (4)). It may even happen that, the method does not work if the Laplace transform does not exist. Another possibility, which was followed in this study is to use a few terms of the Taylor series of *f*. Although the Taylor expansion allowed apply the LT-HPM method, we noted that a possible drawback of this strategy is that it may not produce handy approximate solutions, containing more computational requirements. For comparison purposes, we will consider for both cases study, that the "exact" solution is computed using a scheme based on a trapezoid technique combined with a Richardson extrapolation as a build-in routine from Maple 17. Moreover, the mentioned routine was configured using an absolute error (A.E.) tolerance of 10^{-12}.

In this study was considered the exponential and sine functions respectively and we saw that the process of getting approximate solutions by using LT-HPM, was unnecessarily long and complicated. In order to deal with the above mentioned problems, this paper introduced NDLT-HPM.

At the first place, we studied a nonlinear second order ordinary differential equation with an exponential

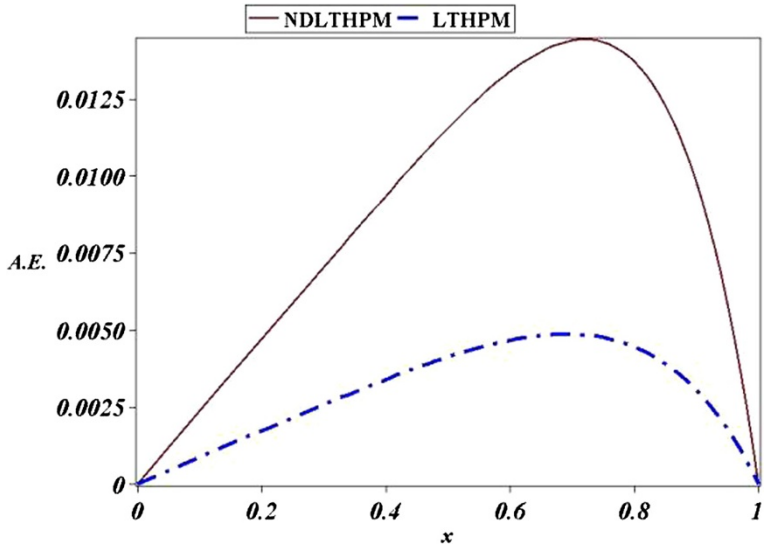

Figure 2 Absolute Error (A.E.) between numerical solution of (21) and LT-HPM, NDLT-HPM approximations (44), (66).

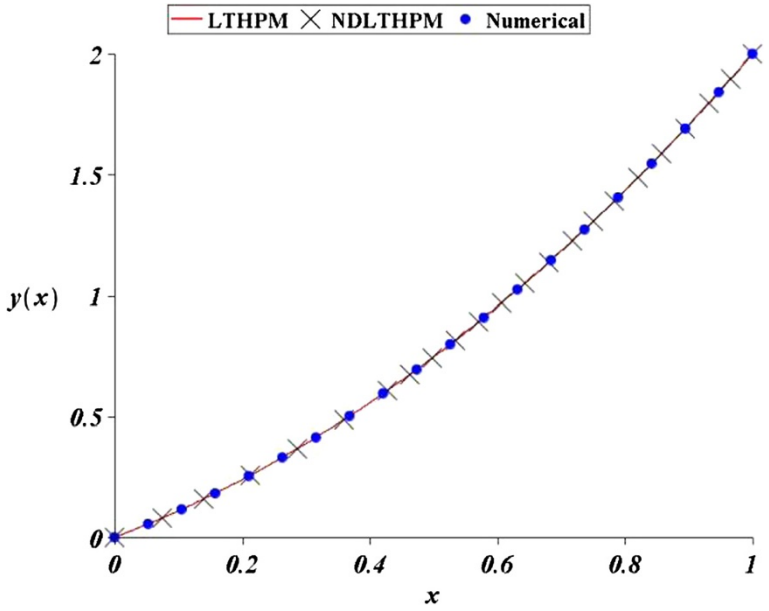

Figure 3 Comparison between numerical solution of (68) and LT-HPM, NDLT-HPM approximations (91), (115).

nonhomogeneous non-polynomial term. This example, proposed the application of LT-HPM, keeping only three terms of the Taylor expansion of e^x from where it was obtained the fourth order approximation (44) and although the final approximation had good accuracy (Figure 1), it is clear that the procedure of solution was cumbersome.

On the other hand, the application of NDLT-HPM to the same problem is outlined in (61), (62), (63), (64) and (65) and can be seen by inspection that exist a considerable saving of computational effort, even NDLT-HPM approximation (66) not only turned out to be clearly shorter than (44), but from the Figures 1 and 2 is scarcely less accurate.

Next, we found an approximate solution for the linear third-order equation of variable coefficients, (68) and although we kept only two terms of the Taylor series of $\sin(x)$, LT-HPM got a precise approximation (see Figure 3). Iterations (86), (87), (88), (89) and (90) for LT-HPM show again a long computational process compared to NDLT-HPM (110), (111), (112), (113) and (114) but the Figure 3 reveals that in fact, both methods are highly accurate, and although NDLT-HPM is handier its absolute error is again slightly less accurate.

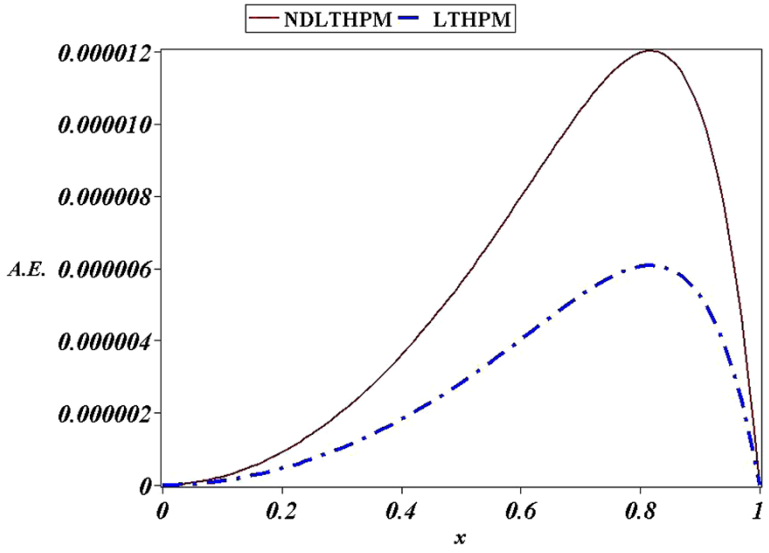

Figure 4 Absolute Error (A.E.) between numerical solution of (68) and LT-HPM, NDLT-HPM approximations (91), (115).

In more precise terms, Figure 2 shows that LT-HPM, NDLT-HPM approximations (44) and (66), are accurate analytical approximate solutions for (21). The biggest absolute error (A.E) of LT-HPM and NDLT-HPM turned out to be 0.000006 and 0.000012 respectively, while from Figure 4 we conclude that the second case study got for the same methods, the values of A.E 0.004 and 0.014. In Spite of this it is noted that NDLT-HPM got a slightly small loss of accuracy with respect to LT-HPM, the comparison of computational effort for both methods leads to the conclusion that NDLT-HPM is more compact, handy and easy to compute, therefore it is an useful tool with good accuracy in the search of solutions for ODES of the type already mentioned.

Finally, we observe that the proposed homotopy formulations (46) and (93) are something different from the original propose in (9) (Vazquez-Leal et al. 2012b), which shows the richness and flexibility of NDHPM and of course of NDLT-HPM. Indeed, the mentioned homotopies were formulated in this way, with the aim that the computational work was reduced considerably, without losing a great precision in the results. Although it is possible to consider other variants of the homotopy given in (9), the key point is that, in the limit when $p \to 1$, the homotopy equation is reduced to the differential equation to be solved.

Conclusions

In this paper NDLT-HPM was introduced as a useful strategy capable of supporting approximate methods, simplifying mathematical iterative procedure, building handy and easy computable expressions in comparison with LT-HPM, in the search for analytical approximate solutions for linear and nonlinear ordinary differential equations with finite boundary conditions, for the case of equations with nonhomogeneous non-polynomial terms. Moreover, the accuracy of the proposed approximate solutions are in good agreement with the exact solutions.

Such as it was explained, NDLT-HPM method expresses the problem of finding an approximate solution for an ordinary differential equation, in terms of solving an algebraic equation for some unknown initial condition (Filobello-Nino et al. 2013). Figure 1 through Figure 4 show how good this procedure is in the search for analytical approximate solutions with good precision, and a moderate computational effort. In addition, just as with LT-HPM, the proposed method does not need to solve several recurrence differential equations. From all the above, we conclude that NDLT-HPM method is a reliable and precise tool in practical applications.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed extensively in the development and completion of this article. All authors read and approved the final manuscript.

Acknowledgments

We gratefully acknowledge the financial support from the National Council for Science and Technology of Mexico (CONACyT) through grant CB-2010-01 #157024. The authors would like to thank Rogelio-Alejandro Callejas-Molina, and Roberto Ruiz-Gomez for their contribution to this project.

Author details

[1]Electronic Instrumentation and Atmospheric Sciences School, Universidad Veracruzana, Circuito Gonzalo Aguirre Beltrán S/N, 9100 Xalapa, Veracruz, México. [2]Higher Colleges of Technology, Abu Dhabi Men's College, P.O. Box 25035, Abu Dhabi, United Arab Emirates. [3]National Institute for Astrophysics, Optics and Electronics, Luis Enrique Erro #1, Sta. María Tonantzintla, 72840 Puebla, México. [4]Department of Artificial Intelligence, Universidad Veracruzana, Sebastián Camacho No. 5 Centro. Xalapa, Veracruz 91000, México. [5]Civil Engineering School, Universidad Veracruzana, Venustiano Carranza S/N, Col. Revolución, 93390 Poza Rica, Veracruz, México.

Received: 13 August 2014 Accepted: 24 September 2014
Published: 9 October 2014

References

Adomian G (1988) A review of decomposition method in applied mathematics. Math Anal Appl 135:501–544

Aminikhah H (2011) Analytical approximation to the solution of nonlinear Blasius viscous flow equation by LTNHPM. Int Sch Res Netw ISRN Math Anal 2012(957473):10, doi:10.5402/2012/957473

Aminikhah H (2012) The combined Laplace transform and new homotopy perturbation method for stiff systems of ODE s. Appl Math Model 36:3638–3644

Aminikhan H, Hemmatnezhad M (2012) A novel effective approach for solving nonlinear heat transfer equations. Heat Transfer Asian Res 41(6):459–466

Araghi MF, Rezapour B (2011) Application of homotopy perturbation method to solve multidimensional schrodingers equations. Int J Math Arch (IJMA) 2:1–6, ISSN 2229–5046

Araghi MF, Sotoodeh M (2012) An enhanced modified homotopy perturbation method for solving nonlinear volterra and fredholm integro-differential equation. World Applied Sciences Journal 20:1646–1655

Assas LMB (2007) Approximate solutions for the generalized K-dV- Burgers' equation by He's variational iteration method. Phys Scr 76:161–164, doi:10.1088/0031-8949/76/2/008

Babolian E, Biazar J (2002) On the order of convergence of Adomian method. Appl Math Comput 130(2):383–387, doi:10.1016/S0096-3003(01)00103-5

Bayat M, Pakar I, Emadi A (2013) Vibration of electrostatically actuated microbeam by means of homotopy perturbation method. Struct Eng Mech 48:823–831

Bayat M, Bayat M, Pakar I (2014) Nonlinear vibration of an electrostatically actuated microbeam. Latin Am J Solids Struct 11:534–544

Belendez A, Pascual C, Alvarez ML, Méndez DI, Yebra MS, Hernández A (2009) High order analytical approximate solutions to the nonlinear pendulum by He's homotopy method. Phys Scr 79(1):1–24, doi:10.1088/0031-8949/79/01/015009

Biazar J, Aminikhan H (2009) Study of convergence of homotopy perturbation method for systems of partial differential equations. Comput Math Appl 58(11–12):2221–2230

Biazar J, Eslami M (2012) A new homotopy perturbation method for solving systems of partial differential equations. Comput Math Appl 62:225–234

Biazar J, Ghanbari B (2012) The homotopy perturbation method for solving neutral functional-differential equations with proportional delays. J King Saud Univ 24:33–37

Biazar J, Ghazvini H (2009) Convergence of the homotopy perturbation method for partial differential equations. Nonlinear Anal 10(5):2633–2640

Chow TL (1995) Classical Mechanics. John Wiley and Sons Inc., New York

Chowdhury SH (2011) A comparison between the modified homotopy perturbation method and Adomian decomposition method for solving nonlinear heat transfer equations. J Appl Sci 11:1416–1420, doi:10.3923/jas.2011.1416.1420

Elias D, Khuri SA, Shishen X (2000) An algorithm for solving boundary value problems. J Comput Phys 159:125–138

El-Shaed M (2005) Application of He's homotopy perturbation method to Volterra's integro differential equation. Int J Nonlinear Sci Numerical Simul 6:163–168

Erdogan U, Ozis T (2011) A smart nonstandard finite difference scheme for second order nonlinear boundary value problems. J Comput Phys 230(17):6464–6474

Evans DJ, Raslan KR (2005) The Tanh function method for solving some important nonlinear partial differential. Int J Computat Math 82:897–905, doi:10.1080/00207160412331336026

Feng X, Mei L, He G (2007) An efficient algorithm for solving Troesch, s problem. Appl Math Comput 189(1):500–507

Fereidon A, Rostamiyan Y, Akbarzade M, Ganji DD (2010) Application of He's homotopy perturbation method to nonlinear shock damper dynamics. Arch Appl Mech 80(6):641–649, doi:10.1007/s00419-009-0334-x

Filobello-Niño U, Vazquez-Leal H, Castañeda-Sheissa R, Yildirim A, Hernandez Martinez L, Pereyra Díaz D, Pérez Sesma A, Hoyos Reyes C (2012) An approximate solution of Blasius equation by using HPM method. Asian J Math Stat :10, doi:10.3923 /ajms.2012, ISSN 1994–5418

Filobello-Nino U, Vazquez-Leal H, Khan Y, Yildirim A, Pereyra-Diaz D, Perez-Sesma A, Hernandez-Martinez L, Sanchez-Orea J, Castaneda-Sheissa R, Rabago-Bernal F (2012) HPM applied to solve nonlinear circuits: a study case. Appl Math Sci 6(85–88):4331–4344

Filobello-Nino U, Vazquez-Leal H, Khan Y, Yildirim A, Jimenez-Fernandez VM, Herrera-May AL, Castaneda-Sheissa R, Cervantes-Perez J (2013) Perturbation method and Laplace–Padé approximation to solve nonlinear problems. Miskolc Math Notes 14(1):89–101

Filobello-Niño U, Vazquez-Leal H, Khan Y, Perez-Sesma A, Diaz-Sanchez A, Herrera-May A, Pereyra-Diaz D, Castañeda-Sheissa R, Jimenez-Fernandez VM, Cervantes-Perez J (2013a) A handy exact solution for flow due to a stretching boundary with partial slip. Revista Mexicana de Física E 59(2013):51–55, ISSN 1870–3542

Filobello-Niño U, Vazquez-Leal H, Boubaker K, Khan Y, Perez-Sesma A, Sarmiento Reyes A, Jimenez-Fernandez VM, Diaz-Sanchez A, Herrera-May A, Sanchez-Orea J, Pereyra-Castro K (2013b) Perturbation method as a powerful tool to solve highly nonlinear problems: the case of Gelfand's Equation. Asian J Math and Stat 2013:7, doi:10.3923 /ajms.2013, ISSN 1994–5418

Filobello-Nino U, Hector V-L, Brahim B, Luis H-M, Yasir K, Jimenez-Fernandez VM, Herrera-May AL, Roberto C-S, Domitilo P-D, Juan C-P, Jose AA P-S, Hernandez-Machuca SF, Leticia C-H (2014) A handy approximation for a mediated bioelectrocatalysis process, related to Michaelis-Menten equation. Springer Plus 3:162, doi:10.1186/2193-1801-3-162

Filobello-Nino U, Vazquez-Leal H, Khan Y, Perez-Sesma A, Diaz-Sanchez A, Jimenez-Fernandez VM, Herrera-May A, Pereyra-Diaz D, Mendez-Perez JM, Sanchez-Orea J (2013) Laplace transform-homotopy perturbation method as a powerful tool to solve nonlinear problems with boundary conditions defined on finite intervals. Comput Appl Math, ISSN: 0101–8205. doi:10.1007/s40314-013-0073-z

Ganji DD, Mirgolbabaei H, Miansari ME, Miansari MO (2008) Application of homotopy perturbation method to solve linear and non-linear systems of ordinary differential equations and differential equation of order three. J Appl Sci 8:1256–1261, doi:10.3923/jas.2008.1256.1261

Hassana HN, El-Tawil MA (2011) An efficient analytic approach for solving two point nonlinear boundary value problems by homotopy analysis method. Math Methods Appl Sci 34:977–989

He JH (1998) A coupling method of a homotopy technique and a perturbation technique for nonlinear problems. Int J Non-Linear Mech 351:37–43, doi:10.1016/S0020-7462(98)00085-7

He JH (1999) Homotopy perturbation technique. Comput Methods Applied Mech Eng 178:257–262, doi:10.1016/S0045-7825(99)00018-3

He JH (2000) A coupling method of a homotopy and a perturbation technique for nonlinear problems. Int J Nonlinear Mech 35(1):37–43

He JH (2006a) Homotopy perturbation method for solving boundary value problems. Phys Lett A 350(1–2):87–88

He JH (2006b) Some asymptotic methods for strongly nonlinear equations. Int J Modern Phys B 20(10):1141–1199, doi:10.1142/S0217979206033796

He JH (2007) Variational approach for nonlinear oscillators. Chaos Solitons Fractals 34:1430–1439, doi:10.1016/j.chaos.2006.10.026

Holmes MH (1995) Introduction to Perturbation Methods. Springer-Verlag, New York

Ji Huan H (2006) Some asymptotic methods for strongly nonlinear equations. Int J Modern Phys B 20(10 (2006)):1141–1199

Kazemnia M, Zahedi SA, Vaezi M, Tolou N (2008) Assessment of modified variational iteration method in BVPs high-order differential equations. J Appl Sci 8:4192–4197, doi:10.3923/jas.2008.4192.4197

Khan Y, Qingbiao W (2011) Homotopy perturbation transform method for nonlinear equations using He's polynomials. Comput Math Appl 61(8):1963–1967

Khan M, Gondal MA, Iqtadar Hussain S, Vanani K (2011) A new study between homotopy analysis method and homotopy perturbation transform method on a semi infinite domain. Math Comput Model 55:1143–1150

King AC, Billingham J, Otto SR (2003) Differential Equations, Linear, Nonlinear, Ordinary, Partial, 1st edn. Cambridge University Press, New York

Kooch A, Abadyan M (2011) Evaluating the ability of modified Adomian decomposition method to simulate the instability of freestanding carbon nanotube: comparison with conventional decomposition method. J Appl Sci 11:3421–3428, doi:10.3923/jas.2011.3421.3428

Kooch A, Abadyan M (2012) Efficiency of modified Adomian decomposition for simulating the instability of nano-electromechanical switches: comparison with the conventional decomposition method. Trends Appl Sci Res 7:57–67, doi:10.3923/tasr.2012.57.67

Madani M, Fathizadeh M, Khan Y, Yildirim A (2011) On the coupling of the homotopy perturbation method and Laplace transformation. Math Comput Model 53(9–10):1937–1945

Mahmoudi J, Tolou N, Khatami I, Barari A, Ganji DD (2008) Explicit solution of nonlinear ZK-BBM wave equation using Exp-function method. J Appl Sci 8:358–363, doi:10.3923/jas.2008.358.363

Marinca V, Herisanu N (2011) Nonlinear Dynamical Systems in Engineering, 1st edn. Springer-Verlag Berlin, Heidelberg

Mirmoradia SH, Hosseinpoura I, Ghanbarpour S, Barari A (2009) Application of an approximate analytical method to nonlinear Troesch, s problem. Appl Math Sci 3(32):1579–1585

Murray R (1988) Spiegel, Teoría y Problemas de Transformadas de Laplace, primera edición. Serie de compendios Schaum, McGraw-Hill, México

Murray JD (2002) Mathematical Biology: I. An Introduction, 3rd edn. Springer-Verlag Berlin Heidelberg, USA, ISBN 0-387-95223-3

Noorzad R, Tahmasebi Poor A, Omidvar M (2008) Variational iteration method and homotopy-perturbation method for solving Burgers equation in fluid dynamics. J Appl Sci 8:369–373, doi:10.3923/jas.2008.369.373

Patel T, Mehta MN, Pradhan VH (2012) The numerical solution of Burger's equation arising into the irradiation of tumour tissue in biological diffusing system by homotopy analysis method. Asian J Appl Sci 5:60–66, doi:10.3923/ajaps.2012.60.66

Rashidi MM, Rastegari MT, Asadi M, Bg OA (2012a) A study of non-newtonian flow and heat transfer over a non-isothermal wedge using the homotopy analysis method. Chem Eng Commun 199:231–256

Rashidi M, Pour SM, Hayat T, Obaidat S (2012b) Analytic approximate solutions for steady flow over a rotating disk in porous medium with heat transfer by homotopy analysis method. Comput Fluids 54:1–9

Sharma PR, Methi G (2011) Applications of homotopy perturbation method to partial differential equations. Asian J Math Stat 4:140–150, doi:10.3923/ajms.2011.140.150

Vanani SK, Heidari S, Avaji M (2011) A low-cost numerical algorithm for the solution of nonlinear delay boundary integral equations. J Appl Sci 11:3504–3509, doi:10.3923/jas.2011.3504.3509

Vazquez-Leal H, Khan Y, Fernández-Anaya G, Herrera-May A, Sarmiento-Reyes A, Filobello-Nino U, Jimenez-Fernández VM, Pereyra-Díaz D (2012a) A general solution for Troesch's problem. Math Probl Eng 208375:14, doi:10.1155/2012/208375

Vazquez-Leal H, Khan Y, Filobello-Nino U, Sarmiento-Reyes A, Diaz-Sanchez A, Cisneros-Sinencio LF (2013) Fixed-Term Homotopy. J Appl Math 2013 (972704):11, doi:10.1155/2013/972704

Vazquez-Leal H, Sarmiento-Reyes A, Khan Y, Filobello-Nino U, Diaz-Sanchez A (2012b) Rational biparameter homotopy perturbation method and Laplace-Padé coupled version. J Appl Math 2012(923975):21, doi:10.1155/2012/923975

Vazquez-Leal H, Filobello-Niño U, Castañeda-Sheissa R, Hernandez Martinez L, Sarmiento-Reyes A (2012c) Modified HPMs inspired by homotopy continuation methods. Math Probl Eng 2012(309123):20, doi:10.155/2012/309123

Vazquez-Leal H, Castañeda-Sheissa R, Filobello-Niño U, Sarmiento-Reyes A, Sánchez-Orea J (2012d) High accurate simple approximation of normal distribution related integrals. Math Probl Eng 2012(124029):22, doi:10.1155/2012/124029

Vazquez-Leal H, Hernandez-Martinez L, Khan Y, Jimenez-Fernandez VM, Filobello-Nino U, Diaz-Sanchez A, Herrera-May AL, Castaneda-Sheissa R, Marin-Hernandez A, Rabago-Bernal F, Huerta-Chua J (2014) Multistage HPM applied to path tracking damped oscillations of a model for HIV infection of CD4+ T cells. British J Math Comput Sci 4(8):1035–1047

Xu F (2007) A generalized soliton solution of the Konopelchenko-Dubrovsky equation using exp-function method. Zeitschrift für Naturforschung - Section A Journal of Physical Sciences 62(12):685–688

Zhang L-N, Xu L (2007) Determination of the limit cycle by He's parameter expansion for oscillators in a potential. Zeitschrift für Naturforschung - Section A Journal of Physical Sciences 62(7–8):396–398

Zill Dennis G (2012) A First Course in Differential Equations with Modeling Applications, 10th edn. Brooks/Cole Cengage Learning, Boston, USA

doi:10.1186/2193-1801-3-594
Cite this article as: Filobello-Nino *et al.*: **Nonlinearities distribution Laplace transform-homotopy perturbation method.** *SpringerPlus* 2014 **3**:594.