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Constructive techniques for zeros of monotone mappings in certain Banach spaces

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Abstract

Let E be a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm, and E^* its dual space. Let $A : E \rightarrow E^*$ be a bounded strongly monotone mapping such that $A^{-1}(0) \neq \emptyset$. For given $x_1 \in E$, let $\{x_n\}$ be generated by the algorithm:

$$x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1,$$

where J is the normalized duality mapping from E into E^* and $\{\alpha_n\}$ is a real sequence in $(0, 1)$ satisfying suitable conditions. Then it is proved that $\{x_n\}$ converges strongly to the unique point $x^* \in A^{-1}(0)$. Finally, our theorems are applied to the convex minimization problem.

Keywords: Strongly monotone, Lipschitz, Bounded

Background

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. An operator $A : H \rightarrow H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle_H \geq 0 \quad \forall x, y \in H, \quad (1.1)$$

and is called *strongly monotone* if there exists $k \in (0, 1)$ such that

$$\langle Ax - Ay, x - y \rangle_H \geq k \|x - y\|_H^2 \quad \forall x, y \in H. \quad (1.2)$$

Interest in monotone operators stems mainly from their usefulness in numerous applications. Consider, for example, the following: Let $f : H \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper lower semi continuous and convex function. The *subdifferential*, $\partial f : H \rightarrow 2^H$ of f at $x \in H$ is defined by

$$\partial f(x) = \{x^* \in H : f(y) - f(x) \geq \langle y - x, x^* \rangle \quad \forall y \in H\}.$$

It is easy to check that $\partial f : H \rightarrow 2^H$ is a *monotone operator* on H , and that $0 \in \partial f(x)$ if and only if x is a minimizer of f . Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of f .

Several existence theorems have been established for the equation $Au = 0$ when A is of the monotone-type (see e.g., Deimling 1985; Pascali and Sburian 1978).

The extension of the monotonicity definition to operators from a Banach space into its dual has been the starting point for the development of nonlinear functional analysis. The monotone maps constitute the most manageable class because of the very simple structure of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts since they can be found in many functional equations. Many of them appear also in calculus of variations as subdifferential of convex functions. (Pascali and Sburian 1978, p. 101).

Let E be a real normed space, E^* its topological dual space. The map $J : E \rightarrow 2^{E^*}$ defined by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|, \|x\| = \|x^*\|\}$$

is called the *normalized duality map* on E . where, \langle, \rangle denotes the generalized duality pairing between E and E^* .

A map $A : E \rightarrow E^*$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq 0. \quad (1.3)$$

A is called *strongly monotone* if there exists $k \in (0, 1)$ such that for each $x, y \in E$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq k\|x - y\|^2. \quad (1.4)$$

A map $A : E \rightarrow E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0. \quad (1.5)$$

A is called *strongly accretive* if there exists $k \in (0, 1)$ such that for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq k\|x - y\|^2. \quad (1.6)$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide. For accretive-type operator A , solutions of the equation $Au = 0$, in many cases, represent *equilibrium state* of some dynamical system (see e.g., Chidume 2009, p. 116).

For approximating a solution of $Au = 0$, assuming existence, where $A : E \rightarrow E$ is of accretive-type, Browder (1967) defined an operator $T : E \rightarrow E$ by $T := I - A$, where I is the identity map on E . He called such an operator *pseudo-contractive*. It is trivial to observe that zeros of A correspond to fixed points of T . For Lipschitz strongly pseudo-contractive maps, Chidume (1987) proved the following theorem.

Theorem C1 (Chidume 1987) *Let $E = L_p$, $2 \leq p < \infty$, and $K \subset E$ be nonempty closed convex and bounded. Let $T : K \rightarrow K$ be a strongly pseudo-contractive and Lipschitz map. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n$, $n \geq 0$, where $\{\lambda_n\} \subset (0, 1)$ satisfies the following conditions:*

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$, (ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T .

By setting $T := I - A$ in Theorem C1, the following theorem for approximating a solution of $Au = 0$ where A is a strongly accretive and bounded operator can be proved.

Theorem C2 Let $E = L_p$, $2 \leq p < \infty$. Let $A : E \rightarrow E$ be a strongly accretive and bounded map. Assume $A^{-1}(0) \neq \emptyset$. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = x_n - \lambda_n Ax_n$, $n \geq 0$, where $\{\lambda_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \lambda_n = \infty$, (ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique solution of $Au = 0$.

The main tool used in the proof of Theorem C1 is an inequality of Bynum (1976). This theorem signalled the return to extensive research efforts on inequalities in Banach spaces and their applications to iterative methods for solutions of nonlinear equations. Consequently, Theorem C1 has been generalized and extended in various directions, leading to flourishing areas of research, for the past thirty years or so, for numerous authors (see e.g., Censor and Reich 1996; Chidume 1986, 1987, 2002; Chidume and Bashir 2007; Chidume and Chidume 2005, 2006; Chidume and Osilike 1999; Deng 1993; Zhou and Jia 1996; Liou 1990; Qihou 1990; Reich 1977, 1978, 1979; Reich and Sabach 2009, 2010; Weng 1991; Xiao 1998; Xu 1991, 1991, 1992; Berinde and Maruster 2014; Moudafi 2003, 2004, 2010; Moudafi and Thera 1997; Xu and Roach 1991; Xu et al. 1995; Zhu 1994 and a host of other authors). Recent monographs emanating from these researches include those by Berinde (2007), Chidume (2009), Goebel and Reich (1984), and William and Shahzad (2014).

Unfortunately, the success achieved in using geometric properties developed from the mid 1980s to early 1990s in approximating zeros of *accretive-type mappings* has not carried over to approximating zeros of *monotone-type operators* in general Banach spaces. Part of the problem is that since A maps E to E^* , for $x_n \in E$, Ax_n is in E^* . Consequently, a recursion formula containing x_n and Ax_n may not be well defined.

Attempts have been made to overcome this difficulty by introducing the inverse of the normalized duality mapping in the recursion formulas for approximating zeros of monotone-type mappings.

In this paper, we introduce an *iterative scheme of Mann-type* to approximate the unique zero of a *strongly monotone bounded mapping* in 2-uniformly convex real Banach with uniformly Gâteaux differentiable norm. Then we apply our results to the convex minimization problem. Finally, our method of proof is of independent interest.

Remark 1 In L_p spaces, $1 < p < \infty$, the formula for J is known precisely (see e.g., Chidume 2009; Cioranescu 1990). In fact, from Cioranescu (1990), Corollary 4.10, p. 72, we have for $J : L_p \rightarrow L_p^*$, $1 < p < \infty$,

$$J(f) = |f|^{p-1} \cdot \text{sign} \frac{f}{\|f\|^{p-1}}.$$

Preliminaries

Let E be a normed linear space. E is said to be smooth if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exist for each $x, y \in S_E$ (Here $S_E := \{x \in E : \|x\| = 1\}$ is the unit sphere of E). E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S_E$, and E is Fréchet differentiable if it is smooth and the limit is attained uniformly for $y \in S_E$.

A normed linear space E is said to be strictly convex if:

$$\|x\| = \|y\| = 1, x \neq y \Rightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

The modulus of convexity of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by:

$$\delta_E(\epsilon) := \inf \left\{ 1 - \frac{1}{2} \|x+y\| : \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon \right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Let $p > 1$. Then E is said to be p -uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Observe that every p -uniformly convex space is uniformly convex.

It is well known that E is smooth if and only if J is single valued. Moreover, if E is a reflexive smooth and strictly convex Banach space, then J^{-1} is single valued, one-to-one, surjective and it is the duality mapping from E^* into E . Finally, if E has uniform Gâteaux differentiable norm, then J is norm-to-weak* uniformly continuous on bounded sets.

In the sequel, we shall need the following results and definitions.

Theorem 2.1 (Xu 1991) *Let $p > 1$ be a given real number. Then the following are equivalent in a Banach space:*

- (i) E is p -uniformly convex.
- (ii) There is a constant $c_1 > 0$ such that for every $x, y \in E$ and $j_x \in J_p(x)$, The following inequality holds:

$$\|x+y\|^p \geq \|x\|^p + p\langle y, j_x \rangle + c_1 \|y\|^p.$$

- (iii) There is a constant $c_2 > 0$ such that for every $x, y \in E$ and $j_x \in J_p(x), j_y \in J_p(y)$, the following inequality holds:

$$\langle x-y, j_x-j_y \rangle \geq c_2 \|x-y\|^p.$$

Corollary 2.2 *Let E be a 2-uniformly convex and smooth real Banach space. Then J^{-1} is Lipschitzian from E^* into E , i.e., there exists constant $L > 0$ such for all $u, v \in E^*$ the following holds*

$$\|J^{-1}u - J^{-1}v\| \leq L\|u - v\|. \quad (2.2)$$

Proof This follows from inequality (iii) of Theorem 2.1 with $p = 2$. \square

Let E be a smooth real Banach space with dual E^* . The function $\phi : E \times E \rightarrow \mathbb{R}$, defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad x, y \in E, \quad (2.3)$$

where J is the normalized duality mapping from E into E^* , introduced by Alber has been studied by Alber (1996), Alber and Guerre-Delabiere (2001), Kamimura and Takahashi (2002), Reich (1979) and a host of other authors. This functional ϕ will play a central role in what follows. If $E = H$, a real Hilbert space, then Eq. (2.3) reduce to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \forall x, y \in E. \quad (2.4)$$

Define a functional $V : E \times E^* \rightarrow \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \quad x \in E, x^* \in E^*. \quad (2.5)$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}x^*) \quad \forall x \in E, x^* \in E^*. \quad (2.6)$$

Lemma 2.3 (Alber 1996) *Let E be a reflexive strictly convex and smooth real Banach space with E^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*) \quad (2.7)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.4 (Kamimura and Takahashi 2002) *Let E be a smooth uniformly convex real Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E . If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 2.5 (Tan and Xu 1993) *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq a_n + \sigma_n \quad n \geq 0.$$

Such that $\sum_{n=0}^{\infty} \sigma_n < \infty$. Then $\lim_{n \rightarrow \infty} a_n$ exists. If addition, the sequence $\{a_n\}$ has a subsequence that converges to 0. Then $\{a_n\}$ converges to 0.

The following results will be useful.

Lemma 2.6 (Alber and Ryazantseva 2006) *For $p > 1$, let X be a p -uniformly convex and smooth real Banach space and S a bounded subset of X . Then there exists a positive constant α such that*

$$\alpha \|x - y\|^p \leq \phi(x, y) \quad \forall x, y \in S.$$

Lemma 2.7 *Let E be a 2-uniformly convex smooth real Banach space. Then the following inequality holds:*

$$\|x - y\|^2 \geq \phi(x, y) + (c_1 - 1)\|x\|^2 \quad x, y \in E,$$

where $0 \leq c_1 \leq 1$ has the same meaning as in Theorem 2.1.

Proof Using (ii) of Theorem 2.1, we have

$$\|x - y\|^2 \geq \|x\|^2 - 2\langle y, Jx \rangle + c_1 \|y\|^2.$$

Interchanging x and y , we obtain

$$\begin{aligned} \|x - y\|^2 &\geq \|y\|^2 - 2\langle x, Jy \rangle + c_1 \|x\|^2 \\ &= \phi(x, y) + (c_1 - 1)\|x\|^2. \end{aligned} \quad \square$$

Main results

We now prove the following result

Theorem 3.1 *Let E be a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm and E^* its dual space. Let $A : E \rightarrow E^*$ be a bounded and k -strongly monotone mapping such that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, let $\{x_n\}$ be the sequence defined iteratively by:*

$$x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1, \quad (3.1)$$

where J is the normalized duality mapping from E into E^* and $\{\alpha_n\} \subset (0, 1)$ is a real sequence satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$. Then, there exists $\gamma_0 > 0$ such that if $\alpha_n < \gamma_0$, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Au = 0$.

Proof The proof is in two steps:

Step 1: We prove that $\{x_n\}$ is bounded. Since $A^{-1}(0) \neq \emptyset$, let $x^* \in A^{-1}(0)$. There exists $r > 0$ such that:

$$r \geq \max \left\{ 4(1 - c_1)\|x^*\|^2, \phi(x_1, x^*) \right\}. \quad (3.2)$$

We show that $\phi(x_n, x^*) \leq r$ for all $n \geq 1$. The proof is by induction. We have $\phi(x_1, x^*) \leq r$. Assume that $\phi(x_n, x^*) \leq r$ for some $n \geq 1$. We show that $\phi(x_{n+1}, x^*) \leq r$. From the induction assumption and Lemma 2.6, there exists $\alpha^* > 0$ such that $\|x_n - x^*\|^2 \leq r\alpha^*$. Since A is bounded, we have:

$$M_0 = 2L \sup\{\|Ax\|^2, \|x - x^*\| \leq \sqrt{r\alpha^*}\} + 1 < \infty, \quad (3.3)$$

where L is a Lipschitz constant of J^{-1} . Define

$$\gamma_0 = \min \frac{1}{2} \left\{ 1, \frac{kr}{M_0} \right\} \quad (3.4)$$

Using the definition of x_{n+1} , we compute as follows:

$$\begin{aligned}\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(Jx_n - \alpha_n Ax_n)) \\ &= V(x^*, Jx_n - \alpha_n Ax_n).\end{aligned}$$

Using Lemma 2.3, with $y^* = \alpha_n Ax_n$, we have:

$$\begin{aligned}\phi(x^*, x_{n+1}) &= V(x^*, Jx_n - \alpha_n Ax_n) \\ &\leq V(x^*, Jx_n) - 2\alpha_n \langle J^{-1}(Jx_n - \alpha_n Ax_n) - x^*, Ax_n - Ax^* \rangle \\ &= \phi(x^*, x_n) - 2\alpha_n \langle x_n - x^*, Ax_n - Ax^* \rangle - 2\alpha_n \langle J^{-1}(Jx_n - \alpha_n Ax_n) - x_n, Ax_n \rangle \\ &= \phi(x^*, x_n) - 2\alpha_n \langle x_n - x^*, Ax_n - Ax^* \rangle \\ &\quad - 2\alpha_n \langle J^{-1}(Jx_n - \alpha_n Ax_n) - J^{-1}(Jx_n), Ax_n \rangle.\end{aligned}$$

Using the strong monotonicity of A , Schwartz inequality and the Lipschitz property of J^{-1} , we obtain

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \phi(x^*, x_n) - 2\alpha_n k \|x_n - x^*\|^2 + 2\alpha_n \|J^{-1}(Jx_n - \alpha_n Ax_n) - J^{-1}(Jx_n)\| \|Ax_n\| \\ &\leq \phi(x^*, x_n) - 2\alpha_n k \|x_n - x^*\|^2 + 2\alpha_n^2 L \|Ax_n\|^2.\end{aligned}$$

Using Lemma 2.7, it follows that

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - 2\alpha_n k \phi(x^*, x_n) + 2\alpha_n k (1 - c_1) \|x^*\|^2 + \alpha_n^2 M_0. \quad (3.5)$$

Finally, using inequality (3.2), the definition of γ_0 (3.4), and the induction assumption, we have

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq (1 - 2k\alpha_n)r + \alpha_n k \frac{r}{2} + \alpha_n k \frac{r}{2} \\ &\leq \left(1 - k\alpha_n \left(2 - \frac{1}{2} - \frac{1}{2}\right)\right)r \\ &\leq (1 - k\alpha_n)r.\end{aligned}$$

Therefore, $\phi(x^*, x_{n+1}) \leq r$. Thus, by induction, $\phi(x^*, x_n) \leq r$ for all $n \geq 1$. So, by inequality (2.4), $\{x_n\}$ is bounded.

Step 2: We now prove that $\{x_n\}$ converges strongly to the unique point x^* of $A^{-1}(0)$. Following the same arguments as in Step 1, using the fact the sequence $\{x_n\}$ is bounded and A is bounded, there exists a positive constant M such that

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - 2\alpha_n k \|x_n - x^*\|^2 + \alpha_n^2 M. \quad (3.6)$$

Therefore,

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) + \alpha_n^2 M.$$

Using the hypothesis $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ and Lemma 2.5, it follows that $\lim_{n \rightarrow \infty} \phi(x^*, x_n)$ exists. From (3.6), we have

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - x^*\| < \infty.$$

Using the fact that $\sum_{n=0}^{\infty} \alpha_n = \infty$, it follows that $\liminf \|x^* - x_n\|^2 = 0$. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow x^*$ as $k \rightarrow \infty$. We have

$$\phi(x^*, x_{n_k}) = \|x\|^2 - 2\langle x^*, Jx_{n_k} \rangle + \|x_{n_k}\|^2.$$

Since $\{x_n\}$ is bounded and J is norm-to weak* uniformly continuous on bounded subsets of E , it follows that $\{\phi(x^*, x_n)\}$ has a subsequence that converges to 0. Thus, by Lemma (), $\{\phi(x^*, x_n)\}$ converges strongly to 0. Applying Lemma(), we obtain that $\|x_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. \square

Corollary 3.2 *Let $E = L_p$, $1 < p \leq 2$ and $A : E \rightarrow E^*$ be a bounded and strongly monotone mapping. For arbitrary $x_1 \in E$, let $\{x_n\}$ be the sequence defined iteratively by:*

$$x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1, \quad (3.7)$$

where J is the normalized duality mapping from E into E^* and $\{\alpha_n\} \subset (0, 1)$ is a real sequence satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Then, there exists $\gamma_0 > 0$ such that if $\alpha_n < \gamma_0$, $\forall n \geq 1$ the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Au = 0$.

Proof Since L_p spaces, $1 < p \leq 2$ are 2-uniformly convex Banach space with uniformly Gâteaux differentiable norm, then the proof follows from Theorem 3.1. \square

Application to convex minimization problems

In this section, we study the problem of finding a minimizer of a convex function f defined from a real Banach space E to \mathbb{R} .

The following basic results are well known.

Lemma 4.1 *Let $f : E \rightarrow \mathbb{R}$ be a real-valued differentiable convex function and $a \in E$. $df : E \rightarrow E^*$ denotes the differential map associated to f . Then the following hold.*

1. *The point a is a minimizer of f on E if and only if $df(a) = 0$.*
2. *If f is bounded, then f is locally Lipschitzian, i.e., for every $x_0 \in E$ and $r > 0$, there exists $\gamma > 0$ such that f is γ -Lipschitzian on $B(x_0, r)$, i.e.*

$$|f(x) - f(y)| \leq \gamma \|x - y\| \quad \forall x, y \in B(x_0, r).$$

Lemma 4.2 *Let E be normed linear space and $f : E \rightarrow \mathbb{R}$ a real-valued differentiable convex function. Assume that f is bounded. Then the differential map $df : E \rightarrow E^*$ is bounded.*

Proof Let $x_0 \in E$ and $r > 0$. Set $B := B(x_0, r)$. We show that $df(B)$ is bounded. From lemma 4.1, there exists $\gamma > 0$ such that

$$|f(x) - f(y)| \leq \gamma \|x - y\| \quad \forall x, y \in B. \quad (4.1)$$

Let $z^* \in df(B)$ and $x^* \in B$ such that $z^* = df(x^*)$. Since B is open, for all $u \in E$, there exists $t > 0$ such that $x^* + tu \in B$. Using the fact that $z^* = df(x^*)$ the convexity of f and inequality (4.1), it follows that

$$\begin{aligned}\langle z^*, tu \rangle &\leq f(x^* + tu) - f(x^*) \\ &\leq t\gamma \|u\|\end{aligned}$$

so that

$$\langle z^*, u \rangle \leq \gamma \|u\| \quad \forall u \in E.$$

Therefore $\|z^*\| \leq \gamma$. Hence $df(B)$ is bounded. \square

Definition 4.3 A function $f : E \rightarrow \mathbb{R}$ is said to be strongly convex if there exists $\alpha > 0$ such that for every $x, y \in E$ with $x \neq y$ and $\lambda \in (0, 1)$, the following inequality holds:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \alpha \|x - y\|^2. \quad (4.2)$$

Lemma 4.4 Let E be normed linear space and $f : E \rightarrow \mathbb{R}$ a real-valued differentiable convex function. Assume that f is strongly convex. Then the differential map $df : E \rightarrow E^*$ is strongly monotone, i.e., there exists a positive constant k such that

$$\langle df(x) - df(y), x - y \rangle \geq k \|x - y\|^2 \quad \forall x, y \in E. \quad (4.3)$$

We now prove the following theorem.

Theorem 4.5 Let E be a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm and let $f : E \rightarrow \mathbb{R}$ be a differentiable, bounded, strongly convex real-valued function which satisfies the growth condition: $f(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$. For arbitrary $x_1 \in E$, let $\{x_n\}$ be the sequence defined iteratively by:

$$x_{n+1} = J^{-1}(Jx_n - \alpha_n df(x_n)), \quad n \geq 1, \quad (4.4)$$

where J is the normalized duality mapping from E into E^* and $\{\alpha_n\} \subset (0, 1)$ is a real sequence satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$. Then, f has a unique minimizer $a^* \in E$ and there exists $\gamma_0 > 0$ such that if $\alpha_n < \gamma_0$, the sequence $\{x_n\}$ converges strongly to a^* .

Proof Since E is reflexive, then from the growth condition, the continuity and the strict convexity of f , f has a unique minimizer a^* characterized by $df(a^*) = 0$ (Lemma 4.1). Finally, from Lemmas 4.2 and 4.4, the differential map $df : E \rightarrow E^*$ is bounded and strongly monotone. Therefore, the proof follows from Theorem 3.1. \square

Conclusion

In this work, we proposed a new iteration scheme for the approximation of zeros of monotone mappings defined in certain Banach spaces. Our results are used to approximate minimizers of convex functions. The results obtained in this paper are important improvements of recent important results in this field.

Authors' contributions

The authors, CD, TMMS, ND and CEC with the consultation of each other carried out this work. All authors read and approved the final manuscript.

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Compliance with ethical guidelines**Competing interests**

The authors declare that they have no competing interests.

Received: 8 June 2015 Accepted: 20 July 2015

Published online: 28 July 2015

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