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Infinite time interval backward stochastic differential equations with continuous coefficients

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Abstract
In this paper, we study the existence theorem for L^p $(1 < p < 2)$ solutions to a class of 1-dimensional infinite time interval backward stochastic differential equations (BSDEs) under the conditions that the coefficients are continuous and have linear growths. We also obtain the existence of a minimal solution. Furthermore, we study the existence and uniqueness theorem for L^p $(1 < p < 2)$ solutions of infinite time interval BSDEs with non-uniformly Lipschitz coefficients. It should be pointed out that the assumptions of this result is weaker than that of Theorem 3.1 in Zong (Turkish J Math 37:704–718, 2013).

Keywords: Backward stochastic, Differential equation (BSDE), Linear growth condition, Comparison theorem

Mathematics Subject Classification: 60H10

Background

The theory of nonlinear backward stochastic differential equations (BSDEs for short) was developed by Pardoux and Peng (1990), from which we know that there exists a unique adapted and square integrable solution to a BSDE of the type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s, \quad t \in [0, T], \tag{1}$$

provided the function g (also called the generator) is Lipschitz in both variables y and z, and ξ and $(g(t, 0, 0))_{0\le t\le T}$ are square integrable. The theory of BSDEs is very useful, due to the connection of this subject with mathematical finance, stochastic control, partial differential equation, stochastic game and stochastic geometry and mathematical economics. Later, many researchers developed the theory of BSDEs and their applications in a series of papers (for example, see Briand et al. (2003), Lepeltier and San Martin (1997), Pardoux (1997, 1998), Karoui et al. (1997) and the references therein) under some other assumptions on the coefficients but for a fixed terminal time $T > 0$. Let us mention the contribution of Lepeltier and San Martin (1997). In Lepeltier and San Martin (1997), the authors got the existence of a solution for a 1-dimensional BSDE where the coefficient

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was continuous, it had linear growth, and the terminal condition was square integrable. They also obtained the existence of a minimal solution.

Chen and Wang (2000) obtained the existence and uniqueness theorem for L^2 solutions of BSDEs with non-uniformly Lipschitz coefficients when $T \equiv \infty$, by the martingale representation theorem and fixed point theorem. In fact, such a problem has been investigated by Peng (1990), Pardoux (1997), Darling and Pardoux (1997) and other researchers under the assumption that terminal value $\xi = 0$ or $E[\mathrm{e}^{p\rho T}|\xi|^p] < \infty$ for some constant ρ and random terminal time T (i.e., T is a stopping time). But in L^p $(1 < p < 2)$, there is no the martingale representation theorem. Zong (2013) studied L^p solutions to infinite time interval BSDEs with non-uniformly Lipschitz coefficients. She gave a new a priori estimate. By using this a priori estimate, she got the existence and uniqueness of L^p solutions to infinite time interval BSDEs.

In this paper, we study the existence theorem for L^p $(1 < p < 2)$ solutions to a class of 1-dimensional infinite time interval BSDEs under the conditions that the coefficients are continuous and have linear growths. We also obtain the existence of a minimal solution. Furthermore, we study the existence and uniqueness theorem for L^p $(1 < p < 2)$ solutions of infinite time interval BSDEs with non-uniformly Lipschitz coefficients. It should be pointed out that the assumptions of this result is weaker than that of Theorem 3.1 in Zong (2013).

This paper is organized as follows. In "Preliminaries" section, we introduce some notations, assumptions and lemmas. In "Main results and proofs" section, we give our main results including the proofs.

Preliminaries

In this section, we shall present some notations, assumptions and lemmas that are used in this paper.

Notation. The Euclidean norm of a vector $x \in R^k$ will be denoted by $|x|$, and for a $k \times d$ matrix A, we define $||A|| = \sqrt{TrAA^*}$, where A^* is the transpose of A.

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space, $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on this space and $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by Brownian motion $(W_t)_{t\geq 0}$, that is

$$\mathcal{F}_t := \sigma\{W_s; s \leq t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets. Furthermore, we define $\mathcal{F} := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$.

We consider the following spaces:

$L^p(\Omega, \mathcal{F}, P, R^k) := \{\xi : \xi$ is R^k-valued and $\mathcal{F}$-measurable random variable such that $E[|\xi|^p] < \infty, p \geq 1\}$;

$\mathcal{L}(\Omega, \mathcal{F}, P, R^k) := \bigcup_{p>1} L^p(\Omega, \mathcal{F}, P, R^k)$;

$\mathcal{S}^p(R^k) := \{V : V_t$ is R^k-valued and $\mathcal{F}_t$-adapted process such that $E[\sup_{t\geq 0} |V_t|^p] < \infty, p \geq 1\}$;

$\mathcal{S}(R^k) := \bigcup_{p>1} \mathcal{S}^p(R^k)$;

$\mathcal{L}^p(R^{k\times d}) := \{V : V_t$ is $R^{k\times d}$-valued and $\mathcal{F}_t$-adapted process such that $E[(\int_0^\infty ||V_s||^2 \mathrm{d}s)^{\frac{p}{2}}] < \infty, p \geq 1\}$;

$\mathcal{L}(R^{k\times d}) := \bigcup_{p>1} \mathcal{L}^p(R^{k\times d})$.

In the sequel, we assume that $1 < p < 2$.

Consider the following infinite time interval BSDE

$$Y_t = \xi + \int_t^\infty g(s, Y_s, Z_s)\mathrm{d}s - \int_t^\infty Z_s \mathrm{d}W_s. \tag{2}$$

Let

$$g : \Omega \times R_+ \times R^k \times R^{k\times d} \mapsto R^k$$

such that for any $(y, z) \in R^k \times R^{k\times d}$, $g(\cdot, y, z)$ is $\mathcal{F}_t$-progressively measurable. We make the following assumptions:

(A.1) There exist two positive non-random functions $\alpha(t)$ and $\beta(t)$, such that for all $y_1, y_2 \in R^k$, $z_1, z_2 \in R^{k\times d}$,

$$\left|g(t, y_1, z_1) - g(t, y_2, z_2)\right| \le \alpha(t)\left|y_1 - y_2\right| + \beta(t)||z_1 - z_2||,$$

where $\alpha(t)$ and $\beta(t)$ satisfy that $\int_0^\infty \alpha(t)\mathrm{d}t < \infty$, $\int_0^\infty \beta^2(t)\mathrm{d}t < \infty$;

(A.1$^{'}$) There exist two positive non-random functions $\alpha(t)$ and $\beta(t)$, such that for all $y_1, y_2 \in R^k$, $z_1, z_2 \in R^{k\times d}$,

$$\left|g(t, y_1, z_1) - g(t, y_2, z_2)\right| \le \alpha(t)\left|y_1 - y_2\right| + \beta(t)||z_1 - z_2||,$$

where $\alpha(t)$ and $\beta(t)$ satisfy that $\int_0^\infty \alpha(t)\mathrm{d}t < \infty$, $\int_0^\infty \beta(t)\mathrm{d}t < \infty$, $\int_0^\infty \beta^2(t)\mathrm{d}t < \infty$;

(A.2) $E\left[\left(\int_0^\infty |g(t, 0, 0)|\mathrm{d}t\right)^p\right] < \infty$;

(A.2$^{'}$) There exists some constant $T \in [0, \infty)$ such that

$$E\left[\left(\int_0^T |g(t, 0, 0)|\mathrm{d}t\right)^p\right] < \infty,$$
$$E\left[\left(\int_T^\infty |g(t, 0, 0)|\mathrm{d}t\right)^2\right] < \infty;$$

(A.3) Linear growth: There exists a positive non-random function $\gamma(t)$ such that

$$|g(\omega, t, y, z)| \le \gamma(t)(1 + |y| + ||z||), \quad \forall(\omega, t, y, z) \in \Omega \times R_+ \times R^k \times R^{k\times d}$$

where $\gamma(t)$ satisfies that $\int_0^\infty \gamma(t)\mathrm{d}t < \infty$, $\int_0^\infty \gamma^2(t)\mathrm{d}t < \infty$;

(A.4) For fixed ω and t, $g(\omega, t, \cdot, \cdot)$ is continuous.

Lemma 1 (see Zong 2013) *Under assumptions* (A.1$^{'}$) *and* (A.2$^{'}$), *if* $\xi \in L^p(\Omega, \mathcal{F}, P, R^k)$, *then BSDE* (2) *has a unique solution* $(Y, Z) \in \mathcal{S}^p(R^k) \times \mathcal{L}^p(R^{k\times d})$.

Main results and proofs

In this section, first we study the existence and uniqueness theorem for L^p solutions of infinite time interval BSDEs with non-uniformly Lipschitz coefficients. It should be pointed out that the assumptions of this result is weaker than that of Lemma 1.

Theorem 2 *Under assumptions (A.1) and (A.2), if* $\xi \in L^p(\Omega, \mathcal{F}, P, R^k)$, *then BSDE* (2) *has a unique solution* $(Y, Z) \in \mathcal{S}^p(R^k) \times \mathcal{L}^p(R^{k\times d})$.

In order to prove Theorem 2, we give an a priori estimate.

Lemma 3 *Suppose that (A.1) holds for g. Furthermore, each ϕ_i $(i = 1, 2)$ satisfies that*

$$E\left[\left(\int_0^\infty |\phi_i(s)|\mathrm{d}s\right)^p\right] < \infty.$$

For any $T \in [0, \infty]$, let $\xi_i \in L^p(\Omega, \mathcal{F}_T, P, R^k)$, $(Y^i, Z^i) \in \mathcal{S}^p(R^k) \times \mathcal{L}^p(R^{k\times d})$ satisfy the following BSDEs:

$$Y_t^i = \xi_i + \int_t^T \left[g\left(s, Y_s^i, Z_s^i\right) + \phi_i(s)\right]\mathrm{d}s - \int_t^T Z_s^i \mathrm{d}W_s, \quad i = 1, 2.$$

Then there exists a positive constant C_p depending only on p such that, for any $\tau \in [0, T]$,

$$\begin{aligned} &E\left[\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^p + \left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ &\quad\le C_p E\left[|\xi_1 - \xi_2|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right] \\ &\qquad + C_p l_{(\tau,T]} E\left[\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^p + \left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right], \end{aligned} \tag{3}$$

where $l_{(\tau,T]} = \left(\int_\tau^T \alpha(s)\mathrm{d}s + \int_\tau^T \beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}} + \left(\int_\tau^T \alpha(s)\mathrm{d}s\right)^p$.

Proof Applying Itô's formula to $\left|Y_t^1 - Y_t^2\right|^2$, we have

$$\begin{aligned} &\left|Y_\tau^1 - Y_\tau^2\right|^2 + \int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s \\ &= |\xi_1 - \xi_2|^2 + 2\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right) + \phi_1(s) - \phi_2(s)\right)\right\rangle \mathrm{d}s \\ &\quad - 2\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right\rangle. \end{aligned} \tag{4}$$

From the Lipschitz assumption (A.1) on *g*, we have

$$\begin{aligned} &2\left\langle Y_s^1 - Y_s^2, \left(g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right)\right)\right\rangle \\ &\quad\le 2\alpha(s)\left|Y_s^1 - Y_s^2\right|^2 + 2\beta(s)\left|Y_s^1 - Y_s^2\right|\left\|Z_s^1 - Z_s^2\right\| \\ &\quad\le 2\alpha(s)\left|Y_s^1 - Y_s^2\right|^2 + 2\beta^2(s)\left|Y_s^1 - Y_s^2\right|^2 + \frac{1}{2}\left\|Z_s^1 - Z_s^2\right\|^2 \\ &\quad\le 2\left(\alpha(s) + \beta^2(s)\right)\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^2 + \frac{1}{2}\left\|Z_s^1 - Z_s^2\right\|^2. \end{aligned} \tag{5}$$

It follows that

$$
\begin{aligned}
&\frac{1}{2}\int_\tau^T \left|\left|Z_s^1 - Z_s^2\right|\right|^2 \mathrm{d}s \\
&\quad \le |\xi_1 - \xi_2|^2 + 2\left(\int_\tau^T \alpha(s)\mathrm{d}s + \int_\tau^T \beta^2(s)\mathrm{d}s\right) \sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^2 \\
&\qquad + 2\int_\tau^T \left|Y_s^1 - Y_s^2\right| |\phi_1(s) - \phi_2(s)|\mathrm{d}s + 2\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right\rangle\right|. \qquad (6)
\end{aligned}
$$

Since $2\int_\tau^T \left|Y_s^1 - Y_s^2\right| |\phi_1(s) - \phi_2(s)|\mathrm{d}s \le \sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^2 + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^2$, we have

$$
\begin{aligned}
&\int_\tau^T \left|\left|Z_s^1 - Z_s^2\right|\right|^2 \mathrm{d}s \\
&\quad \le 4\left(|\xi_1 - \xi_2|^2 + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^2\right) \\
&\qquad + 4\left(1 + \int_\tau^T \alpha(s)\mathrm{d}s + \int_\tau^T \beta^2(s)\mathrm{d}s\right) \sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^2 \\
&\qquad + 4\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right\rangle\right|. \qquad (7)
\end{aligned}
$$

Using the following fact: if b, $a_i \ge 0$ and $b \le \sum_{i=1}^n a_i$, then $b^p \le \sum_{i=1}^n a_i^p$ for any $p \in (0,1)$, we have

$$
\begin{aligned}
&\left(\int_\tau^T \left|\left|Z_s^1 - Z_s^2\right|\right|^2 \mathrm{d}s\right)^{\frac{p}{2}} \\
&\quad \le c_p\left(|\xi_1 - \xi_2|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right) \\
&\qquad + c_p\left[1 + \left(\int_\tau^T \alpha(s)\mathrm{d}s + \int_\tau^T \beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}}\right] \sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^p \\
&\qquad + c_p\left(\left|\int_\tau^T (Y_s^1 - Y_s^2)(Z_s^1 - Z_s^2)\mathrm{d}W_s\right|^{\frac{p}{2}}\right), \qquad (8)
\end{aligned}
$$

where c_p is a positive constant depending only on p. By the Burkholder–Davis–Gundy inequality, we get

$$
\begin{aligned}
&c_pE\left[\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right\rangle\right|^{\frac{p}{2}}\right] \\
&\quad \le d_pE\left[\left(\int_\tau^T \left|Y_s^1 - Y_s^2\right|^2 \left|\left|Z_s^1 - Z_s^2\right|\right|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right] \\
&\quad \le d_pE\left[\sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^{\frac{p}{2}} \left(\int_\tau^T \left|\left|Z_s^1 - Z_s^2\right|\right|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right] \qquad (9)
\end{aligned}
$$

and thus

$$c_pE\left[\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right\rangle\right|^{\frac{p}{2}}\right] \le \frac{1}{2}E\left[\left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] + \frac{d_p^2}{2}E\left[\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^p\right], \tag{10}$$

where d_p is a positive constant depending only on p. From (8) and (10), we have

$$\begin{aligned} &E\left[\left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ &\le C\left(E[|\xi_1 - \xi_2|^p] + E\left[\left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right]\right) \\ &\quad + C\big(1 + l_{(\tau,T]}\big)E\left[\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^p\right], \end{aligned} \tag{11}$$

where C is a positive constant depending only on p.

On the other hand, we prove

$$\begin{aligned} &E\left[\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^p\right] \\ &\le C'E\left[|\xi_1 - \xi_2|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right] \\ &\quad + C' l_{(\tau,T]}E\left[\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^p + \left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right], \end{aligned} \tag{12}$$

where C' is a positive constant depending only on p. Obviously, $\left\{\int_\tau^t \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s; \tau \le t \le T\right\}$ is an $\mathcal{F}_t$-martingale. Thus, it follows that

$$Y_t^1 - Y_t^2 = E\left[(\xi_1 - \xi_2) + \int_t^T \left(g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right) + \phi_1(s) - \phi_2(s)\right)\mathrm{d}s|\mathcal{F}_t\right]. \tag{13}$$

Applying Doob's inequality, we can deduce that

$$\begin{aligned} &E\left[\sup_{t\in[\tau,T]}\left|Y_t^1 - Y_t^2\right|^p\right] \\ &\le E\left[\sup_{t\in[\tau,T]}\left(E\left[|\xi_1 - \xi_2| + \int_\tau^T \left|g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right) + \phi_1(s) - \phi_2(s)\right|\mathrm{d}s|\mathcal{F}_t\right]\right)^p\right] \\ &\le \left(\frac{p}{p-1}\right)^p E\left[\left(|\xi_1 - \xi_2| + \int_\tau^T \left|g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right) + \phi_1(s) - \phi_2(s)\right|\mathrm{d}s\right)^p\right] \\ &\le D_pE\left[|\xi_1 - \xi_2|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p + \left(\int_\tau^T \left|g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right)\right|\mathrm{d}s\right)^p\right], \end{aligned} \tag{14}$$

where D_p is a positive constant depending only on p. From the Lipschitz assumption (A.1) on g, we have

$$\begin{aligned}
&E\left[\left(\int_\tau^T \left|g\left(s,Y_s^1,Z_s^1\right)-g\left(s,Y_s^2,Z_s^2\right)\right|\mathrm{d}s\right)^p\right]\\
&\quad\le E\left[\left(\int_\tau^T \left(\alpha(s)\left|Y_s^1-Y_s^2\right|+\beta(s)\left\|Z_s^1-Z_s^1\right\|\right)\mathrm{d}s\right)^p\right]\\
&\quad\le M_p\left(\int_\tau^T \alpha(s)\mathrm{d}s\right)^p E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p\right]\\
&\qquad+M_p\left(\int_\tau^T \beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}} E\left[\left(\int_\tau^T \left\|Z_s^1-Z_s^2\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right],
\end{aligned}\tag{15}$$

where M_p is a positive constant depending only on p. From (14) and (15), we have

$$\begin{aligned}
&E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p\right]\\
&\quad\le C^{'}E\left[|\xi_1-\xi_2|^p+\left(\int_0^\infty |\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right]\\
&\qquad+C^{'}l_{(\tau,T]}E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p+\left(\int_\tau^T \left\|Z_s^1-Z_s^2\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right],
\end{aligned}\tag{16}$$

where $C^{'}$ is a positive constant depending only on p.

Combining (11) with (16), we get

$$\begin{aligned}
&E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p+\left(\int_\tau^T \left|Z_s^1-Z_s^2\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\quad\le C_pE\left[|\xi_1-\xi_2|^p+\left(\int_0^\infty |\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right]\\
&\qquad+C_pl_{(\tau,T]}E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p+\left(\int_\tau^T \left\|Z_s^1-Z_s^2\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right],
\end{aligned}\tag{17}$$

where C_p is a positive constant depending only on p. The proof of Lemma 3 is complete. □

Proof of Theorem 2 Let $\xi^n:=(\xi\wedge n)\vee(-n)$ and $g_n(t,y,z):=g(t,y,z)-g(t,0,0)$ $+h_n(g(t,0,0))$, where $h_n(g(t,0,0)):=\frac{g(t,0,0)ne^{-t}}{|g(t,0,0)|\vee(ne^{-t})}$. By Theorem 1.2 in Chen and Wang (2000), BSDE

$$Y_t^n=\xi^n+\int_t^\infty g_n\left(s,Y_s^n,Z_s^n\right)\mathrm{d}s-\int_t^\infty Z_s^n\mathrm{d}W_s$$

has a unique solution $(Y^n,Z^n)\in\mathcal{S}^2(R^k)\times\mathcal{L}^2(R^{k\times d})$. Since

$$\left(\int_0^\infty \alpha(s)\mathrm{d}s + \int_0^\infty \beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}} + \left(\int_0^\infty \alpha(s)\mathrm{d}s\right)^p < \infty,$$

we can choose a strictly increasing sequence $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = \infty$, such that

$$l_{(t_i,t_{i+1}]} \leq \frac{1}{2C_p}, \quad i = 0, 1, 2, \ldots, N.$$

Applying Lemma 3, we have

$$\begin{aligned} &E\left[\sup_{s\in[t_i,t_{i+1}]} |Y_s^{m+n} - Y_s^n|^p + \left(\int_{t_i}^{t_{i+1}} ||Z_s^{m+n} - Z_s^n||^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ &\quad \leq C_p E\left[\left|Y_{t_{i+1}}^{m+n} - Y_{t_{i+1}}^n\right|^p\right] \\ &\qquad + C_p E\left[\left(\int_0^\infty |h_{n+m}(g(s,0,0)) - h_n(g(s,0,0))|\mathrm{d}s\right)^p\right] \\ &\qquad + \frac{1}{2} E\left[\sup_{s\in[t_i,t_{i+1}]} |Y_s^{m+n} - Y_s^n|^p + \left(\int_{t_i}^{t_{i+1}} ||Z_s^{m+n} - Z_s^n||^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]. \end{aligned} \tag{18}$$

Thus

$$\begin{aligned} &E\left[\sup_{s\in[t_i,t_{i+1}]} |Y_s^{m+n} - Y_s^n|^p + \left(\int_{t_i}^{t_{i+1}} ||Z_s^{m+n} - Z_s^n||^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ &\quad \leq 2C_p E\left[\left|Y_{t_{i+1}}^{m+n} - Y_{t_{i+1}}^n\right|^p\right] \\ &\qquad + 2C_p E\left[\left(\int_0^\infty |h_{n+m}(g(s,0,0)) - h_n(g(s,0,0))|\mathrm{d}s\right)^p\right] \\ &\quad \leq 2C_p E\left[\sup_{s\in[t_{i+1},t_{i+2}]} |Y_s^{m+n} - Y_s^n|^p + \left(\int_{t_{i+1}}^{t_{i+2}} ||Z_s^{m+n} - Z_s^n||^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ &\qquad + 2C_p E\left[\left(\int_0^\infty |h_{n+m}(g(s,0,0)) - h_n(g(s,0,0))|\mathrm{d}s\right)^p\right], \quad i = 0, 1, 2, \ldots, N-1. \end{aligned} \tag{19}$$

In particular, we have

$$\begin{aligned} &E\left[\sup_{s\geq t_N} |Y_s^{m+n} - Y_s^n|^p + \left(\int_{t_N}^\infty ||Z_s^{m+n} - Z_s^n||^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ &\quad \leq 2C_p E\left[|\xi^{m+n} - \xi^n|^p\right] \\ &\qquad + 2C_p E\left[\left(\int_0^\infty |h_{n+m}(g(s,0,0)) - h_n(g(s,0,0))|\mathrm{d}s\right)^p\right]. \end{aligned} \tag{20}$$

From (19) and (20), it follows that

$$
\begin{aligned}
&E\left[\sup_{s\geq 0}|Y_s^{n+m}-Y_s^n|^p+\left(\int_0^\infty ||Z_s^{n+m}-Z_s^n||^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\leq \sum_{i=0}^{N}E\left[\sup_{s\in[t_i,t_{i+1}]}|Y_s^{m+n}-Y_s^n|^p+\left(\int_{t_i}^{t_{i+1}}||Z_s^{m+n}-Z_s^n||^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\leq (2C_p+(2C_p)^2+\cdots+(2C_p)^{N+1})E\left[|\xi^{m+n}-\xi^n|^p\right]\\
&\quad+(N+1)(2C_p+(2C_p)^2+\cdots+(2C_p)^{N+1})E\left[\left(\int_0^\infty |h_{n+m}(g(s,0,0))-h_n(g(s,0,0))|\mathrm{d}s\right)^p\right]\\
&\leq \overline{C}E\left[|\xi^{m+n}-\xi^n|^p\right]\\
&\quad+\overline{C}E\left[\left(\int_0^\infty |h_{n+m}(g(s,0,0))-h_n(g(s,0,0))|\mathrm{d}s\right)^p\right],
\end{aligned}
\tag{21}
$$

where $\overline{C}=(N+1)(2C_p+(2C_p)^2+\cdots+(2C_p)^{N+1})$. The right-hand side of Inequality (21) clearly tends to 0, as $n\to\infty$, uniformly in m, so we have a Cauchy sequence and the limit is a solution to BSDE (2). Let us consider (Y,Z) and (Y',Z') to be two solutions to BSDE (2). In a similar manner to the proof of Inequality (21), we can obtain

$$
E\left[\sup_{s\geq 0}\left|Y_s-Y_s'\right|^p+\left(\int_0^\infty \left|\left|Z_s-Z_s'\right|\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\leq 0.
$$

Thus, we get immediately $(Y,Z)=(Y',Z')$. The proof of Theorem 2 is complete. □

Theorem 4 (Comparison Theorem) *Assume that $k=1$. We make the same assumptions on ξ, g and $\overline{\xi}, \overline{g}$ as in Theorem 2. Let $(\overline{Y},\overline{Z})$ be a solution of BSDE*

$$
\overline{Y}_t=\overline{\xi}+\int_t^\infty \overline{g}(s,\overline{Y}_s,\overline{Z}_s)\mathrm{d}s-\int_t^\infty \overline{Z}_s\mathrm{d}W_s.
$$

If we suppose that:

$$
\hat{\xi}:=\xi-\overline{\xi}\leq 0,\quad \hat{g}_t:=g(t,\overline{Y}_t,\overline{Z}_t)-\overline{g}(t,\overline{Y}_t,\overline{Z}_t)\leq 0,\ \text{ a.s.,}
\tag{22}
$$

then

$$
\hat{Y}_t:=Y_t-\overline{Y}_t\leq 0,\ \text{ a.s., }\ \forall t\in[0,\infty).
$$

Moreover, $\overline{Y}_t=Y_t$ a.s., if and only if $\overline{\xi}=\xi$ a.s., $\overline{g}(t,Y_t,Z_t)=g(t,Y_t,Z_t)$ a.s..

Proof Suppose that $W_t=(W_t^1,W_t^2,\ldots,W_t^d)^T$, $\forall t\in[0,\infty)$, where W_t^i is the ith components of W_t. Let us consider the following BSDEs

$$
\begin{aligned}
Y_t&=\xi+\int_t^\infty g(s,Y_s,Z_s)\mathrm{d}s-\int_t^\infty Z_s\mathrm{d}W_s,\\
\overline{Y}_t&=\overline{\xi}+\int_t^\infty \overline{g}(s,\overline{Y}_s,\overline{Z}_s)\mathrm{d}s-\int_t^\infty \overline{Z}_s\mathrm{d}W_s,
\end{aligned}
$$

where $Z_t = (Z_t^1, Z_t^2, \ldots, Z_t^d)^T$, $\overline{Z}_t = (\overline{Z}_t^1, \overline{Z}_t^2, \ldots, \overline{Z}_t^d)^T$, $\forall t \in [0, \infty)$ and $\int_t^\infty Z_s \mathrm{d}W_s = \sum_{i=1}^d \int_t^\infty Z_t^i \mathrm{d}W_t^i$, $\int_t^\infty \overline{Z}_s \mathrm{d}W_s = \sum_{i=1}^d \int_t^\infty \overline{Z}_t^i \mathrm{d}W_t^i$. Then, we have

$$\hat{Y}_t = \hat{\xi} + \int_t^\infty \left(a_s \hat{Y}_s + \left\langle b_s, \hat{Z}_s \right\rangle + \hat{g}_s \right) \mathrm{d}s - \int_t^\infty \hat{Z}_s \mathrm{d}W_s, \tag{23}$$

where

$$\begin{aligned}
\hat{Z}_s &= Z_s - \overline{Z}_s = (Z_t^1 - \overline{Z}_t^1, Z_t^2 - \overline{Z}_t^2, \ldots, Z_t^d - \overline{Z}_t^d)^T, \\
Z_s^{(i)} &= (\overline{Z}_s^1, \ldots, \overline{Z}_s^i, Z_s^{i+1}, \ldots, Z_s^d)^T, \quad i = 1, 2, \ldots, d-1, \\
Z_s^{(0)} &= Z_s = (Z_s^1, Z_s^2, \ldots, Z_s^d)^T, \\
Z_s^{(d)} &= \overline{Z}_s = (\overline{Z}_s^1, \overline{Z}_s^2, \ldots, \overline{Z}_s^d)^T, \\
a_s &= \frac{g(s, Y_s, Z_s) - g\left(s, \overline{Y}_s, Z_s\right)}{\hat{Y}_s} 1_{\left\{\hat{Y}_s \neq 0\right\}}, \\
b_s^i &= \frac{g\left(s, \overline{Y}_s, Z_s^{(i-1)}\right) - g\left(s, \overline{Y}_s, Z_s^{(i)}\right)}{Z_s^i - \overline{Z}_s^i} 1_{\left\{Z_s^i - \overline{Z}_s^i \neq 0\right\}}, \quad i = 1, 2, \ldots, d, \\
b_s &= (b_s^1, b_s^2, \ldots, b_s^d)^T,
\end{aligned}$$

which imply $|a_s| \leq \alpha(s), |b_s| \leq \beta(s)$.

Solving (23), we know that the unique solution of BSDEs (23) can be represented as

$$\hat{Y}_t = E\left[\hat{\xi} X_\infty + \int_t^\infty \hat{g}_s X_s \mathrm{d}s | \mathcal{F}_t\right], \tag{24}$$

where

$$X_s = \exp\left[\int_t^s \left(a_r - \frac{1}{2}|b_r|^2\right) \mathrm{d}r + \int_t^s b_r \mathrm{d}W_r\right], \quad s \geq t.$$

From (24), we can obtain $\hat{Y}_t \leq 0$, a.s. and if $\overline{\xi} = \xi$ a.s., $\overline{g}(t, Y_t, Z_t) = g(t, Y_t, Z_t)$ a.s., then $\overline{Y}_t = Y_t$ a.s..

Choosing $t = 0$ in (24) and from the strict monotonicity of $E[\cdot]$, we can obtain that if $\overline{Y}_0 = Y_0$, then $\overline{\xi} = \xi$ a.s., $\overline{g}(t, Y_t, Z_t) = g(t, Y_t, Z_t)$ a.s.. The proof of Theorem 4 is complete. □

Now we prove the existence theorem for L^p solutions of 1-dimensional infinite time interval BDSDEs which generalizes Theorem 1 in Lepeltier and San Martin (1997).

Theorem 5 *Assume that $k = 1$. Under assumptions (A.3) and (A.4), if $\xi \in L^p(\Omega, \mathcal{F}, P, R)$, then BSDE (2) has a solution $(Y, Z) \in \mathcal{S}^p(R) \times \mathcal{L}^p(R^d)$. Also, there is a minimal solution $(\underline{Y}, \underline{Z})$ of BSDE (2), in the sense that for any other solution (Y, Z) of (2), we have $\underline{Y} \leq Y$.*

In order to prove Theorem 5, we need the following lemmas.

Lemma 6 *Suppose that (A.3) and (A.4) hold for g. For each $(\omega, t, y, z) \in \Omega \times R_+ \times R \times R^d$, define the sequence of functions*

$$g_n(\omega,t,y,z) := \inf_{\left(y',z'\right)\in Q}\left\{g(\omega,t,y',z') + n\gamma(t)\left(\left|y-y'\right| + \left|z-z'\right|\right)\right\},$$

where Q is the set of all rational numbers in R^{d+1}. *Then* g_n *satisfies*

(i) *Linear growth:* $\forall(\omega,t,y,z) \in \Omega \times R_+ \times R \times R^d, |g_n(\omega,t,y,z)| \le \gamma(t)(1+|y|+|z|)$;
(ii) *Monotonicity in n:* $\forall(\omega,t,y,z) \in \Omega \times R_+ \times R \times R^d$, $g_n(\omega,t,y,z) \uparrow$;
(iii) *Lipschitz condition:* $\forall(\omega,t,y,z),(\omega,t,y',z') \in \Omega \times R_+ \times R \times R^d$,

$$\left|g_n(\omega,t,y,z) - g_n(\omega,t,y',z')\right| \le n\gamma(t)\left(\left|y-y'\right| + \left|z-z'\right|\right);$$

(iv) *Strong convergence: if* $(y_n,z_n) \to (y,z)$, *as* $n \to \infty$, *then*

$$g_n(\omega,t,y_n,z_n) \to g(\omega,t,y,z), \text{ as } n \to \infty.$$

The proof of Lemma 6 is very similar to that of Lemma 1 in Lepeltier and San Martin (1997), so we omit it.

We also define the function

$$G(\omega,t,y,z) := \gamma(t)(1+|y|+|z|), \quad \forall(\omega,t,y,z) \in \Omega \times R_+ \times R \times R^d.$$

For each given $\xi \in L^p(\Omega,\mathcal{F},P,R)$, by Theorem 2, there exist two pair of processes (Y^n,Z^n) and (U,V), which are the solutions to the following BSDEs

$$Y_t^n = \xi + \int_t^\infty g_n(s,Y_s^n,Z_s^n)\mathrm{d}s - \int_t^\infty Z_s^n \mathrm{d}W_s, \tag{25}$$

$$U_t = \xi + \int_t^\infty G(s,U_s,V_s)\mathrm{d}s - \int_t^\infty V_s \mathrm{d}W_s, \tag{26}$$

respectively. From Theorem 4 and Lemma 6, we get

$$\forall n \ge m, \ Y^m \le Y^n \le U, \ \text{a.s.} \tag{27}$$

Lemma 7 *There exists a constant* $A > 0$ *independent of n, such that*

$$E\left[\sup_{t\ge 0}|U_t|^p\right] \le A, \quad E\left[\left(\int_0^\infty |V_t|^2\mathrm{d}t\right)^{\frac{p}{2}}\right] \le A,$$
$$E\left[\sup_{t\ge 0}\left|Y_t^n\right|^p\right] \le A, \quad E\left[\left(\int_0^\infty \left|Z_t^n\right|^2\mathrm{d}t\right)^{\frac{p}{2}}\right] \le A, \quad \forall n \in N.$$

Proof Since (U,V) is the solution of BSDE (26), there exists a constant $B > 0$ independent of n, such that

$$E\left[\sup_{t\ge 0}|U_t|^p\right] \le B, \quad E\left[\left(\int_0^\infty |V_t|^2\mathrm{d}t\right)^{\frac{p}{2}}\right] \le B.$$

From Inequality (27), we can obtain that for each $n \in N$,

$$|Y_t^n|^p \leq 2^{p-1}\Big(|Y_t^1|^p + |U_t|^p\Big).$$

Thus, there exists a constant $C > 0$ independent of n, such that

$$E\left[\sup_{t\geq 0}|Y_t^n|^p\right] \leq C, \quad \forall n \in N.$$

At last, we prove the boundedness of $E\left[\left(\int_0^\infty |Z_t^n|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right]$. Applying Itô's formula to $|Y_t^n|^2$, we have

$$\begin{aligned}&|Y_0^n|^2 + \int_0^\infty |Z_t^n|^2 \mathrm{d}t \\ &\quad = |\xi|^2 + 2\int_0^\infty Y_t^n g_n(t, Y_t^n, Z_t^n)\mathrm{d}t - 2\int_0^\infty Y_t^n Z_t^n \mathrm{d}W_t. \end{aligned} \tag{28}$$

By Lemma 6 (i), we know $|g_n(t,y,z)| \leq \gamma(t)(1+|y|+|z|)$. Thus, we have

$$\begin{aligned} 2|Y_t^n g_n(t, Y_t^n, Z_t^n)| &\leq 2\gamma(t)\Big(|Y_t^n| + |Y_t^n|^2 + |Y_t^n Z_t^n|\Big) \\ &\leq \gamma(t)\Big(1+|Y_t^n|^2\Big) + 2\gamma(t)|Y_t^n|^2 \\ &\quad + 2\gamma^2(t)|Y_t^n|^2 + \frac{1}{2}|Z_t^n|^2 \\ &\leq \gamma(t) + 3\Big(\gamma(t)+\gamma^2(t)\Big)\sup_{t\geq 0}|Y_t^n|^2 + \frac{1}{2}|Z_t^n|^2 \end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}\int_0^\infty |Z_t^n|^2 \mathrm{d}t &\leq 2|\xi|^2 + 2\int_0^\infty \gamma(t)\mathrm{d}t \\ &\quad + 6\left(\int_0^\infty \gamma(t)\mathrm{d}t + \int_0^\infty \gamma^2(t)\mathrm{d}t\right)\sup_{t\geq 0}|Y_t^n|^2 + 4\left|\int_0^\infty Y_t^n Z_t^n \mathrm{d}W_t\right|. \end{aligned} \tag{30}$$

Using the following fact: if $b,\ a_i \geq 0$ and $b \leq \sum_{i=1}^n a_i$, then $b^p \leq \sum_{i=1}^n a_i^p$ for any $p \in (0,1)$, we have

$$\begin{aligned}\left(\int_0^\infty |Z_t^n|^2 \mathrm{d}t\right)^{\frac{p}{2}} &\leq c_p\left(|\xi|^p + \left(\int_0^\infty \gamma(t)\mathrm{d}t\right)^{\frac{p}{2}}\right) \\ &\quad + c_p\left(\int_0^\infty \gamma(t)\mathrm{d}t + \int_0^\infty \gamma^2(t)\mathrm{d}t\right)^{\frac{p}{2}} \\ &\quad \sup_{t\geq 0}|Y_t^n|^p + c_p\left|\int_0^\infty Y_t^n Z_t^n \mathrm{d}W_t\right|^{\frac{p}{2}}, \end{aligned} \tag{31}$$

where c_p is a positive constant depending only on p. By the Burkholder–Davis–Gundy inequality, we get

$$c_pE\left[\left|\int_0^\infty Y_t^n Z_t^n \mathrm{d}W_t\right|^{\frac{p}{2}}\right] \leq d_pE\left[\left(\int_0^\infty \left|Y_t^n Z_t^n\right|^2 \mathrm{d}t\right)^{\frac{p}{4}}\right]$$
$$\leq d_pE\left[\sup_{t\geq 0}\left|Y_t^n\right|^{\frac{p}{2}}\left(\int_0^\infty \left|Z_t^n\right|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right] \tag{32}$$

and thus

$$c_pE\left[\left|\int_0^\infty Y_t^n Z_t^n \mathrm{d}W_t\right|^{\frac{p}{2}}\right] \leq \frac{1}{2}E\left[\left(\int_0^\infty \left|Z_t^n\right|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right] + \frac{d_p^2}{2}E\left[\sup_{t\geq 0}\left|Y_t^n\right|^p\right], \tag{33}$$

where d_p is a positive constant depending only on p. From (31) and (33), we have

$$E\left[\left(\int_0^\infty \left|Z_t^n\right|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right]$$
$$\leq C_p\left[E[|\xi|^p] + \left(\int_0^\infty \gamma(t)\mathrm{d}t\right)^{\frac{p}{2}}\right]$$
$$+ C_p\left[1 + \left(\int_0^\infty \gamma(t)\mathrm{d}t + \int_0^\infty \gamma^2(t)\mathrm{d}t\right)^{\frac{p}{2}}\right]E\left[\sup_{t\geq 0}\left|Y_t^n\right|^p\right], \tag{34}$$

where C_p is a positive constant depending only on p. Thus, there exists a constant $A > 0$ independent of n, such that

$$E\left[\sup_{t\geq 0}|U_t|^p\right] \leq A, \quad E\left[\left(\int_0^\infty |V_t|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right] \leq A,$$
$$E\left[\sup_{t\geq 0}\left|Y_t^n\right|^p\right] \leq A, \quad E\left[\left(\int_0^\infty \left|Z_t^n\right|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right] \leq A, \ \forall n \in N.$$

The proof of Lemma 7 is complete. □

Lemma 8 *$\{(Y^n, Z^n)\}_{n=1}^\infty$ converges in $\mathcal{S}^p(R) \times \mathcal{L}^p(R^d)$.*

Proof Since $\{Y^n\}_{n=1}^\infty$ is increasing and bounded in $\mathcal{S}^p(R)$, we deduce from the dominated convergence theorem that Y^n converges in $\mathcal{S}^p(R)$. We shall denote by Y the limit of Y^n. Applying Itô's formula to $\left|Y_t^n - Y_t^m\right|^2$, we get for any $n, m \in N$,

$$\left|Y_0^n - Y_0^m\right|^2 + \int_0^\infty \left|Z_t^n - Z_t^m\right|^2 \mathrm{d}t$$
$$= 2\int_0^\infty \left(Y_t^n - Y_t^m\right)\left(g_n\left(t, Y_t^n, Z_t^n\right) - g_m\left(t, Y_t^m, Z_t^m\right)\right)\mathrm{d}t$$
$$- 2\int_0^\infty \left(Y_t^n - Y_t^m\right)\left(Z_t^n - Z_t^m\right)\mathrm{d}W_t. \tag{35}$$

Thus, we have

$$\begin{aligned}
&\int_0^\infty |Z_t^n - Z_t^m|^2 \mathrm{d}t \\
&\leq 2\sup_{t\geq 0}|Y_t^n - Y_t^m| \int_0^\infty |g_n(t, Y_t^n, Z_t^n)| \mathrm{d}t + 2\sup_{t\geq 0}|Y_t^n - Y_t^m| \int_0^\infty |g_m(t, Y_t^m, Z_t^m)| \mathrm{d}t \\
&\quad + 2\left|\int_0^\infty (Y_t^n - Y_t^m)(Z_t^n - Z_t^m) \mathrm{d}W_t\right|.
\end{aligned} \tag{36}$$

Using the following fact: if b, $a_i \geq 0$ and $b \leq \sum_{i=1}^n a_i$, then $b^p \leq \sum_{i=1}^n a_i^p$ for any $p \in (0,1)$, it follows that

$$\begin{aligned}
&E\left[\left(\int_0^\infty |Z_t^n - Z_t^m|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right] \\
&\leq c_p E\left[\sup_{t\geq 0}|Y_t^n - Y_t^m|^{\frac{p}{2}} \left(\int_0^\infty |g_n(t, Y_t^n, Z_t^n)| \mathrm{d}t\right)^{\frac{p}{2}}\right] \\
&\quad + c_p E\left[\sup_{t\geq 0}|Y_t^n - Y_t^m|^{\frac{p}{2}} \left(\int_0^\infty |g_m(t, Y_t^m, Z_t^m)| \mathrm{d}t\right)^{\frac{p}{2}}\right] \\
&\quad + c_p E\left[\left|\int_0^\infty (Y_t^n - Y_t^m)(Z_t^n - Z_t^m) \mathrm{d}W_t\right|^{\frac{p}{2}}\right],
\end{aligned} \tag{37}$$

where c_p is a positive constant depending only on p. From Schwarz's inequality, we have

$$\begin{aligned}
&E\left[\sup_{t\geq 0}|Y_t^n - Y_t^m|^{\frac{p}{2}} \left(\int_0^\infty \left|g_k\left(t, Y_t^k, Z_t^k\right)\right| \mathrm{d}t\right)^{\frac{p}{2}}\right] \\
&\leq \left(E\left[\sup_{t\geq 0}|Y_t^n - Y_t^m|^p\right]\right)^{\frac{1}{2}} \left(E\left[\left(\int_0^\infty \left|g_k\left(t, Y_t^k, Z_t^k\right)\right| \mathrm{d}t\right)^p\right]\right)^{\frac{1}{2}}, \quad k = n, m.
\end{aligned} \tag{38}$$

By Lemma 6 (i), we can obtain

$$\begin{aligned}
&E\left[\left(\int_0^\infty \left|g_k\left(t, Y_t^k, Z_t^k\right)\right| \mathrm{d}t\right)^p\right] \\
&\leq E\left[\left(\int_0^\infty \gamma(t)\left(1 + \left|Y_t^k\right| + \left|Z_t^k\right|\right) \mathrm{d}t\right)^p\right] \\
&\leq d_p\left(\int_0^\infty \gamma(t) \mathrm{d}t\right)^p + d_p\left(\int_0^\infty \gamma(t) \mathrm{d}t\right)^p E\left[\sup_{t\geq 0}\left|Y_t^k\right|^p\right] \\
&\quad + d_p\left(\int_0^\infty \gamma^2(t) \mathrm{d}t\right)^{\frac{p}{2}} E\left[\left(\int_0^\infty \left|Z_t^k\right|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right], \quad k = n, m
\end{aligned} \tag{39}$$

where d_p is a positive constant depending only on p. Thus, by Lemma 7, there exists a constant $D > 0$ independent of n, m such that

$$E\left[\sup_{t\geq 0}|Y_t^n - Y_t^m|^{\frac{p}{2}} \left(\int_0^\infty |g_n(t, Y_t^n, Z_t^n)| \mathrm{d}t\right)^{\frac{p}{2}}\right] \leq D\left(E\left[\sup_{t\geq 0}|Y_t^n - Y_t^m|^p\right]\right)^{\frac{1}{2}}, \tag{40}$$

$$E\left[\sup_{t\geq 0}\left|Y_t^n - Y_t^m\right|^{\frac{p}{2}}\left(\int_0^\infty \left|g_m\left(t, Y_t^m, Z_t^m\right)\right|\mathrm{d}t\right)^{\frac{p}{2}}\right] \leq D\left(E\left[\sup_{t\geq 0}\left|Y_t^n - Y_t^m\right|^p\right]\right)^{\frac{1}{2}}. \quad (41)$$

By the Burkholder–Davis–Gundy inequality, we get

$$\begin{aligned} c_pE&\left[\left|\int_0^\infty \left(Y_t^n - Y_t^m\right)\left(Z_t^n - Z_t^m\right)\mathrm{d}W_t\right|^{\frac{p}{2}}\right] \\ &\leq e_pE\left[\left(\int_0^\infty \left|Y_t^n - Y_t^m\right|^2\left|Z_t^n - Z_t^m\right|^2\mathrm{d}t\right)^{\frac{p}{4}}\right] \\ &\leq e_pE\left[\sup_{t\geq 0}\left|Y_t^n - Y_t^m\right|^{\frac{p}{2}}\left(\int_0^\infty \left|Z_t^n - Z_t^m\right|^2\mathrm{d}t\right)^{\frac{p}{4}}\right] \end{aligned} \quad (42)$$

and thus

$$\begin{aligned} c_pE\left[\left|\int_0^\infty \left(Y_t^n - Y_t^m\right)\left(Z_t^n - Z_t^m\right)\mathrm{d}W_t\right|^{\frac{p}{2}}\right] \leq \frac{1}{2}E\left[\left(\int_0^\infty \left|Z_t^n - Z_t^m\right|^2\mathrm{d}t\right)^{\frac{p}{2}}\right] \\ + \frac{e_p^2}{2}E\left[\sup_{t\geq 0}\left|Y_t^n - Y_t^m\right|^p\right], \end{aligned} \quad (43)$$

where e_p is a positive constant depending only on p. From (37), (40), (41) and (43), we have

$$\begin{aligned} E&\left[\left(\int_0^\infty \left|Z_t^n - Z_t^m\right|^2\mathrm{d}t\right)^{\frac{p}{2}}\right] \\ &\leq C_p\left(\left(E\left[\sup_{t\geq 0}\left|Y_t^n - Y_t^m\right|^p\right]\right)^{\frac{1}{2}} + E\left[\sup_{t\geq 0}\left|Y_t^n - Y_t^m\right|^p\right]\right), \end{aligned} \quad (44)$$

where C_p is a positive constant depending only on p. Thus, $\{Z^n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{L}^p(R^d)$, from which the result follows. The proof of Lemma 8 is complete.

Proof of Theorem 5 For all $n \in N$, we have $Y^n \leq U$, and $\{Y^n\}_{n=1}^\infty$ converges in $\mathcal{S}^p(R)$, $\mathrm{d}t \times \mathrm{d}P$-a.s. to $Y \in \mathcal{S}^p(R)$.

On the other hand, since Z^n converges in $\mathcal{L}^p(R^d)$ to Z, we can assume, choosing a subsequence if needed, that $Z^n \to Z$, $\mathrm{d}t \times \mathrm{d}P$-a.s., as $n \to \infty$ and $\overline{G} := \sup_n |Z^n|$ is $\mathrm{d}t \times \mathrm{d}P$ integrable. Therefore, by Lemma 6 (i) and (iv), we get for almost all ω,

$$\begin{aligned} &g_n\left(t, Y_t^n, Z_t^n\right) \to g(t, Y_t, Z_t), \quad \mathrm{d}t - \text{a.e.}, \quad \text{as } n \to \infty, \\ &\left|g_n\left(t, Y_t^n, Z_t^n\right)\right| \leq \gamma(t)\left(1 + \left|Y_t^n\right| + \left|Z_t^n\right|\right) \\ &\qquad \leq \gamma(t)\left(1 + \sup_n\left|Y_t^n\right| + \overline{G}_t\right) \in L^1([0,\infty); \mathrm{d}t). \end{aligned} \quad (45)$$

Thus, for almost all ω and uniformly in t, it holds that

$$\int_t^\infty g_n\big(s, Y_s^n, Z_s^n\big)\mathrm{d}s \to \int_t^\infty g(s, Y_s, Z_s)\mathrm{d}s, \quad \text{as } n \to \infty.$$

From the continuity properties of the stochastic integral, it follows that

$$\sup_{t\geq 0}\left|\int_t^\infty Z_s^n \mathrm{d}W_s - \int_t^\infty Z_s \mathrm{d}W_s\right| \to 0 \text{ in probability}, \quad \text{as } n \to \infty.$$

Choosing again, a subsequence, we can assume that the above convergence is P-a.s. Finally,

$$\begin{aligned}\big|Y_t^n - Y_t^m\big| \leq &\int_t^\infty \big|g_n\big(s, Y_s^n, Z_s^n\big) - g_m\big(s, Y_s^m, Z_s^m\big)\big|\mathrm{d}s \\ &+ \left|\int_t^\infty \big(Z_s^n - Z_s^m\big)\mathrm{d}W_s\right|,\end{aligned} \tag{46}$$

and taking limits on m and supremum over t, we get

$$\begin{aligned}\sup_{t\geq 0}\big|Y_t^n - Y_t\big| \leq &\int_0^\infty \big|g_n\big(s, Y_s^n, Z_s^n\big) - g(s, Y_s, Z_s)\big|\mathrm{d}s \\ &+ \sup_{t\geq 0}\left|\int_t^\infty \big(Z_s^n - Z_s\big)\mathrm{d}W_s\right|, \ P - \text{a.s.}\end{aligned} \tag{47}$$

from which it follows that Y^n converges uniformly in t to Y (in particular, Y is a continuous process). Note that $\{Y^n\}_{n=1}^\infty$ is monotone; therefore, we actually have the uniform convergence for the entire sequence and not just for a subsequence. Taking limits in Equation (25), we deduce that (Y, Z) is a solution of BSDE (2).

Let $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}^p(R) \times \mathcal{L}^p(R^d)$ be any solution of BSDE (2). From Theorem 4, we get that $Y^n \leq \tilde{Y}$, $\forall n \in N$ and therefore $Y \leq \tilde{Y}$ proving that Y is the minimal solution. The proof of Theorem 5 is complete. □

Remark 9 By Theorem 5, we have: Under the assumption (A.3) and (A.4), for each given $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P, R)$, BSDE (2) has a solution $(Y, Z) \in \mathcal{S}(R) \times \mathcal{L}(R^d)$. Also, in $\mathcal{S}(R) \times \mathcal{L}(R^d)$, there is a minimal solution $(\underline{Y}, \underline{Z})$ of BSDE (2), in the sense that for any other solution (Y, Z) of (2), we have $\underline{Y} \leq Y$.

Conclusion

In this paper, we have solved two problems on infinite time interval BSDEs. Firstly, by using an a priori estimate (Lemma 3), we studied the existence and uniqueness theorem for $L^p\,(1 < p < 2)$ solutions of infinite time interval BSDEs with non-uniformly Lipschitz coefficients (Theorem 2). It should be pointed out that the assumptions of Theorem 2 is weaker than that of Theorem 3.1 in Zong (2013). Secondly, applying comparison theorem for 1-dimensional infinite time interval BSDEs (Theorem 4), we studied the existence theorem for $L^p\,(1 < p < 2)$ solutions of 1-dimensional infinite time interval BSDEs under the conditions that the coefficients are continuous and have linear growths (Theorem 5). In Theorem 5, the existence of a minimal solution was also obtained.

Authors' contributions

The authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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