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Parallel algorithm for convection–diffusion system based on least-squares procedure

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Abstract

Combining subspace correction method with least-squares finite element procedure, we construct a new overlapping domain decomposition parallel algorithm for solving the first-order time-dependent convection–diffusion system. This algorithm is fully parallel. We analyze the convergence of approximate solution, and study the dependence of the convergent rate on the spacial mesh size, time increment, iteration number and sub-domains overlapping degree. Both theoretical analysis and numerical results suggest that only one or two iterations are needed to reach to given accuracy at each time step.

Keywords: Overlapping domain decomposition, Parallel subspace correction, Least-squares, Convection–diffusion system

Mathematics Subject Classification: 65M55, 65M60, 65M12, 65M15

Background

In this paper, we consider the following initial-boundary value problem for time-dependent convection–diffusion system:

$$\begin{cases} c \frac{\partial u}{\partial t} + \nabla \cdot \sigma + qu = f, & x \in \Omega, \quad 0 < t \leq T, \\ \sigma + A \nabla u + \mathbf{b}u = 0, & x \in \Omega, \quad 0 < t \leq T, \\ u = 0, & x \in \Gamma_D, \\ \sigma \cdot \nu = 0, & x \in \Gamma_N, \quad 0 \leq t \leq T, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1)$$

where Ω is an open bounded domain \mathbf{R}^d ($1 \leq d \leq 3$), with a Lipschitz continuous boundary $\Gamma = \Gamma_D \cup \Gamma_N$; and ν is the unit vector normal to Γ_N ; the flow field $\mathbf{b} = (b_1, b_2, \dots, b_d)^T$; the source term $q = q(x, t) \geq 0$ and exterior flow function $f = f(x, t)$ are some given functions; the coefficient $c = c(x)$ is positive function and the diffusion coefficient matrix $A = (a(i, j))_{d \times d}$ is a symmetric uniformly positive definite matrix, i.e., there exist some positive constants c_* and a_* such that

$$a_* \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j, \quad c_* \leq c(x), \quad \forall \xi \in \mathbf{R}^d, x \in \Omega. \quad (2)$$

This type of partial differential equation arises in many important fields, such as the mathematical modeling of aerodynamics, porous medium fluid flow, fluid dynamics (e.g.

Euler equations, Navier-Stokes equations), meteorology, and semiconductor devices. Many numerical methods have been established to simulate this problem, for example, finite element and finite difference method, Eulerian–Lagrangian localized adjoint method Celia et al. (1990). The streamline diffusion finite element method Hughes and Brooks (1979), least-squares mixed element methods Yang (1999, 2000, 2002), Zhang and Guo (2012), Zhang et al. (2011) and Zhang (2009), and so on. Generally, these numerical procedures result in a large scale of algebraic system, so it is very important and useful to develop effective parallel algorithms both in engineering applications and mathematical analysis.

Recently domain decomposition parallel computation has become a powerful tool for solving a large scale system of partial differential equations. A lot of work has been done on domain decomposition parallel algorithms, for example, see Beilina (2016), Bramble et al. (1990, 1991), Cai (1989), Dolean et al. (2008, 2015), Dryja and Widlund (1987), Lu et al. (1991), Ma et al. (2009), Tarek (2008), Xu (1989, 1992, 2001) and Yang (2010). But many parallel algorithms based on overlapping domain decomposition are iterative algorithms so that many iteration steps are needed to reach given accuracy, which leads to much more global amount of computational work. On the basis of the idea of the parallel subspace correction method proposed by Xu (1989, 1992, 2001), the first author of this paper and Yang established a new parallel algorithm combined with characteristic finite element scheme, finite difference scheme and least-square scheme for one dimensional convection–diffusion problem in Zhang et al. (2011) and Zhang and Yang (2011a, b), where both theoretical analysis and numerical results suggest that when overlapping degree has a positive lower bound independent of mesh size, only one or two iterative times is needed to reach the optimal convergence precision at each time level.

In this paper, using the same technique as in Zhang et al. (2011), Zhang and Yang (2011) and Zhang and Yang (2011), we establish a new parallel algorithm for solving the convection–diffusion system. Here the arbitrary dimensional problem is considered, unlike in Zhang and Yang (2011) only one dimensional model was studied. And the different least-squares finite element scheme from the one in Zhang and Yang (2011) is used to obtain the optimal L^2 -norm error estimate. The partition of unity is applied to distribute the corrections in the overlapping domains reasonably in this parallel algorithm. We analyze the convergence of approximate solution, and study the dependence of the convergent rate on the spacial mesh size, time increment, iteration number and sub-domains overlapping degree. Both theoretical analysis and numerical experiments indicate the full parallelization of the algorithms and very good approximate property.

Parallel algorithm

Throughout this paper we use usual definitions and notations of Sobolev spaces as in Adams (1975). Let $W^{k,p}(\Omega)$ ($k \geq 0$, $1 \leq p \leq \infty$) be Sobolev spaces defined on Ω with usual norms $\|\cdot\|_{W^{k,p}(\Omega)}$ and $H^k(\Omega) = W^{k,2}(\Omega)$. Define inner products as follows:

$$(u, v) = \int_{\Omega} u(x)v(x)dx \quad \forall u, v \in L^2(\Omega),$$

$$(\sigma, \omega) = \sum_{i=1}^d (\sigma_i, \omega_i) \quad \forall \sigma, \omega \in [L^2(\Omega)]^d, \quad 1 \leq d \leq 3.$$

Introduce the spaces $\mathcal{W} = \{\omega \in [L^2(\Omega)]^d; \nabla \cdot \omega \in L^2(\Omega), \omega \cdot \nu = 0 \text{ on } \Gamma_N\}$ and $\mathcal{V} = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_D\}$. Make a time partition $0 = t_0 < t_1 < \dots < t_{M-1} < t_M = T$ and set $\tau_n = t_n - t_{n-1}$ and $\tau = \max_{1 \leq n \leq M} \tau_n$. Let $w^n(x) = w(x, t_n)$. By use of the difference technique with first-order accuracy to discretize the first-order system (1), we can rewrite the system (1) as follows [see Yang (1999)]

$$\begin{aligned} c(x)\bar{\partial}_t u^n(x) + \nabla \cdot \sigma^n(x) + q^n(x)u^n(x) &= f^n(x) + R^n(x), \quad x \in \Omega, \\ \bar{\partial}_t \sigma^n(x) + A(x)\nabla \bar{\partial}_t u^n(x) + \bar{\partial}_t(\mathbf{b}^n(x)u^n(x)) &= 0, \quad x \in \Omega, \\ u^n(x) &= 0, \quad x \in \Gamma_D, \\ \sigma^n(x) \cdot \nu(x) &= 0, \quad x \in \Gamma_N, \\ u^0(x) &= u_0(x), \quad x \in \Omega \end{aligned} \tag{3}$$

where

$$\begin{aligned} R^n(x) &= c(x)(\bar{\partial}_t u^n(x) - u_t(x)) = O\left(\tau_n \frac{\partial^2 u}{\partial t^2}\right), \\ \bar{\partial}_t u^n(x) &= (u^n - u^{n-1})/\tau_n. \end{aligned}$$

To construct parallel subspace correction algorithm, we firstly make a domain decomposition. Assume that $\{\Omega'_i\}_{i=1}^N$ is a non-overlapping domain decomposition of Ω . In order to obtain an overlapping domain decomposition, we extend each subregion Ω'_i to a larger region Ω_i such that $\Omega'_i \subset \Omega_i \subset \Omega$ and $dist(\partial \Omega'_i \setminus \partial \Omega, \partial \Omega_i \setminus \partial \Omega) \geq H$ for each $1 \leq i \leq N$, where $H > 0$ is called as overlapping degree. Let \mathcal{T}_{h_u} and \mathcal{T}_{h_σ} be two families of quasi-regular finite element partitions of the domain Ω such that the elements in the partitions have the diameters bounded by h_u and h_σ , respectively. Assume that $\mathcal{T}_{h_u, i} = \mathcal{T}_{h_u} \cap \Omega_i$ and $\mathcal{T}_{h_\sigma, i} = \mathcal{T}_{h_\sigma} \cap \Omega_i$ just are one finite element partition of Ω_i for $1 \leq i \leq N$. Let $\mathcal{W}_{h_\sigma} \subset \mathcal{W}$, and $\mathcal{V}_{h_u} \subset \mathcal{V}$ be piecewise r -degree and k -degree polynomial spaces defined on the partitions \mathcal{T}_{h_σ} and \mathcal{T}_{h_u} respectively.

Denote by \tilde{A} the inverse of A and define a bilinear form

$$\begin{aligned} a_n((\sigma, w), (\omega, v)) &= \left(\frac{1}{c}(c w + \tau_n(\nabla \cdot \sigma + q^n w)), c v + \tau_n(\nabla \cdot \omega + q^n v)\right) \\ &\quad + \tau_n(\tilde{A}(\sigma + A \nabla w + \mathbf{b}^n w), \omega + A \nabla v + \mathbf{b}^n v). \end{aligned}$$

Based on (3) and Yang (1999), we get the standard least-squares finite element procedure:

Least-squares scheme Given an initial approximation $(\varrho_h^0, w_h^0) \in \mathcal{W}_{h_\sigma} \times \mathcal{V}_{h_u}$. For $n = 1, 2, \dots, M$, seek $(\varrho_h^n, w_h^n) \in \mathcal{W}_{h_\sigma} \times \mathcal{V}_{h_u}$ such that

$$\begin{aligned} a_n((\varrho_h^n, w_h^n), (\omega_h, v_h)) &= \left(\frac{1}{c}(c w_h^{n-1} + \tau_n f^n), c v_h + \tau_n(\nabla \cdot \omega_h + q^n v_h)\right) \\ &\quad + \tau_n(\tilde{A}(\varrho_h^{n-1} + A \nabla w_h^{n-1} + \mathbf{b}^n w_h^{n-1}), \omega_h + A \nabla v_h + \mathbf{b}^n v_h), \\ \forall (\omega_h, v_h) &\in \mathcal{W}_{h_\sigma} \times \mathcal{V}_{h_u}. \end{aligned} \tag{4}$$

In the following part of this section, we propose the parallel domain decomposition algorithm of the system (4). Define finite element sub-spaces:

$$\mathcal{V}_{h_u}^i = \{v_h \in \mathcal{V}_{h_u}; v_h = 0 \text{ in } \Omega \setminus \Omega_i\}, \quad 1 \leq i \leq N$$

and

$$\mathcal{W}_{h_\sigma}^i = \{\sigma_h \in \mathcal{W}_{h_\sigma}; \sigma_h = 0 \text{ in } \Omega \setminus \Omega_i\}, \quad 1 \leq i \leq N.$$

It is clear that

$$\mathcal{V}_{h_u} = \mathcal{V}_{h_u}^1 + \mathcal{V}_{h_u}^2 + \dots + \mathcal{V}_{h_u}^N$$

and

$$\mathcal{W}_{h_\sigma} = \mathcal{W}_{h_\sigma}^1 + \mathcal{W}_{h_\sigma}^2 + \dots + \mathcal{W}_{h_\sigma}^N.$$

It is easily seen that there exists a finite open covering family $\{O^i\}_{i=1}^N$ of the domain Ω such that $O^i \cap \Omega \subset \Omega_i$. We know that there exists a partition of unity $\{\varphi_i\}_{i=1}^N$ (see Toselli and Widlund (2005), Lemma 3.4) such that

- (a) $\text{supp}(\varphi_i) \subset O^i, 0 \leq \varphi_i \leq 1, \|\varphi_i\|_{W^{r,\infty}} \leq CH^{-r}, 1 \leq i \leq N;$
- (b) $\varphi_1 + \varphi_2 + \dots + \varphi_N = 1 \text{ in } \Omega.$

Let $\varphi_{h_u}^i$ and $\varphi_{h_\sigma}^i$ be the nodal piecewise linear interpolation of φ_i on the finite element meshes \mathcal{T}_{h_u} and \mathcal{T}_{h_σ} , and \mathcal{I}_{h_u} and \mathcal{I}_{h_σ} be the interpolating operators on \mathcal{V}_{h_u} and \mathcal{W}_{h_σ} .

Based on (4), we formulate the parallel subspace correction algorithm.

Parallel algorithm Let m denote the iteration number at each time step. Give an initial approximation $(\sigma_h^0, u_h^0) = (\varrho_h^0, w_h^0) \in \mathcal{W}_{h_\sigma} \times \mathcal{V}_{h_u}$. For $n = 1, 2, \dots, M$, seek $(\sigma_h^n, u_h^n) \in \mathcal{W}_{h_\sigma} \times \mathcal{V}_{h_u}$ by four steps:

Step 1. Set $(\tilde{\sigma}_0^n, \tilde{u}_0^n) = (\sigma_h^{n-1}, u_h^{n-1})$ and $j := 1$.

Step 2. For $i = 1, 2, \dots, N$, seek $(\varepsilon_j^i, e_j^i) \in \mathcal{W}_{h_\sigma}^i \times \mathcal{V}_{h_u}^i$, in parallel, such that

$$\begin{aligned} & a_n \left((\varepsilon_j^i, e_j^i), (\omega_h, v_h) \right) \\ &= \left(\frac{1}{c} (cu_h^{n-1} + \tau_n f^n), c\mathcal{I}_{h_u}(\varphi_{h_u}^i v_h) \right. \\ & \quad \left. + \tau_n (\nabla \cdot \mathcal{I}_{h_\sigma}(\varphi_{h_\sigma}^i \omega_h) + q^n \mathcal{I}_{h_u}(\varphi_{h_u}^i v_h)) \right) \\ & \quad + \tau_n \left(\tilde{A}(\sigma_h^{n-1} + A\nabla u_h^{n-1} + \mathbf{b}^n u_h^{n-1}), \mathcal{I}_{h_\sigma}(\varphi_{h_\sigma}^i \omega_h) \right) \\ & \quad + A\nabla \mathcal{I}_{h_u}(\varphi_{h_u}^i v_h) + \mathbf{b}^n \mathcal{I}_{h_u}(\varphi_{h_u}^i v_h) \\ & \quad - a_n \left((\tilde{\sigma}_{j-1}^n, \tilde{u}_{j-1}^n), (\mathcal{I}_{h_\sigma}(\varphi_{h_\sigma}^i \omega_h), \mathcal{I}_{h_u}(\varphi_{h_u}^i v_h)) \right), \\ & \quad \forall (\omega_h, v_h) \in \mathcal{W}_{h_\sigma} \times \mathcal{V}_{h_u}. \end{aligned} \tag{5}$$

Step 3. Set corrections

$$\tilde{\sigma}_j^n = \tilde{\sigma}_{j-1}^n + \sum_{i=1}^N \varepsilon_j^i, \quad \tilde{u}_j^n = \tilde{u}_{j-1}^n + \sum_{i=1}^N e_j^i. \tag{6}$$

Step 4. If $j < m$, then set $j := j + 1$ and return the step 2; or set

$$\sigma_h^n = \tilde{\sigma}_m^n, \quad u_h^n = \tilde{u}_m^n$$

and then return back to the first step to start iteration at the next time step.

Some lemmas and main result

In the following sections, we denote by K and δ some general constants and small positive constants independent of the mesh parameters H, h_σ, h_u and τ , which may be different at different occurrences. Let

$$\begin{aligned} \|(\omega, v)\|_{a_n}^2 &= \left(\frac{1}{c} (cv + \tau_n(\nabla \cdot \omega + q^n v)), cv + \tau_n(\nabla \cdot \omega + q^n v) \right) \\ &\quad + \tau_n \left(\tilde{A}(\omega + A\nabla v + \mathbf{b}^n v), \omega + A\nabla v + \mathbf{b}^n v \right). \end{aligned}$$

In order to analyze the convergence of parallel algorithm, we introduce projection operators $P_{h_\sigma}^i : \mathcal{W}_{h_\sigma} \rightarrow \mathcal{W}_{h_\sigma}^i$ and $Q_{h_u}^i : \mathcal{V}_{h_u} \rightarrow \mathcal{V}_{h_u}^i$ such that

$$\begin{aligned} a_n \left((P_{h_\sigma}^i \omega, Q_{h_u}^i v), (\omega_h, v_h) \right) &= a_n((\omega, v), (\omega_h, v_h)), \\ \forall (\omega_h, v_h) \in \mathcal{W}_{h_\sigma}^i \times \mathcal{V}_{h_u}^i \quad i &= 1, 2, \dots, N. \end{aligned}$$

Now, we give some important lemmas which are used to analyze the convergence of parallel algorithm.

We assume that finite element spaces \mathcal{W}_{h_σ} and \mathcal{V}_{h_u} have the inverse property and approximate properties [see Ciarlet (1978)] that there exist some integers $r, r_1, k > 0$, such that, for $1 \leq q \leq \infty$ and $\forall \omega \in H(\text{div}; \Omega) \cap [W^{r+1,q}(\Omega)]^d$,

$$\begin{aligned} \inf_{\omega_h \in \mathcal{W}_{h_\sigma}} \|\omega - \omega_h\|_{[L^q(\Omega)]^d} &\leq Kh_\sigma^{r+1} \|\omega\|_{[W^{r+1,q}(\Omega)]^d}, \\ \inf_{\omega_h \in \mathcal{W}_{h_\sigma}} \|\nabla \cdot (\omega - \omega_h)\|_{L^q(\Omega)} &\leq Kh_\sigma^{r_1} \|\nabla \cdot \omega\|_{W^{r_1,q}(\Omega)}, \\ \inf_{v_h \in \mathcal{V}_{h_u}} \|v - v_h\|_{L^q(\Omega)} &\leq Kh_u^{k+1} \|v\|_{W^{k+1,q}(\Omega)}, \quad \forall v \in L^2(\Omega) \cap W^{k+1,q}(\Omega). \end{aligned}$$

Based on Theorem 3.3 in Yang (1999), the following result can be read:

Lemma 1 *Let (σ, u) and (ϱ_h^n, w_h^n) be the solutions of (1) and least-squares scheme, respectively. Then there holds the a priori error estimate*

$$\max_n \|u^n - w_h^n\|_{L^2(\Omega)} + \max_n \|\sigma^n - \varrho_h^n\|_{[L^2(\Omega)]^d} \leq K\{h_u^{k+1} + h_\sigma^{r_1} + \tau\}. \tag{7}$$

Lemma 2 [See Yang (2001)] *For any function $\varphi \in W^{1,\infty}(\Omega)$ and $\omega_h \in \mathcal{W}_{h_\sigma}$, we have the following estimate*

$$\|\varphi\omega_h - \mathcal{I}_{h_\sigma}(\varphi\omega_h)\|_{[L^2(\Omega)]^d} \leq Kh_\sigma \min(\|\varphi\|_{W^{1,\infty}(\Omega)} \|\omega_h\|_{[L^2(\Omega)]^d}, \|\varphi\|_{H^1(\Omega)} \|\omega_h\|_{[L^\infty(\Omega)]^d}),$$

where $d = 1, 2, 3$.

Lemma 3 For $1 \leq i \leq N$, we have

$$\begin{aligned} \|(\mathcal{I} - \mathcal{I}_{h_\sigma})(\varphi_{h_\sigma}^i \omega_h)\|_{[L^2(\Omega)]^d} &\leq K \frac{h_\sigma}{H} \|\omega_h\|_{[L^2(\Omega)]^d}, \quad \forall \omega_h \in \mathcal{W}_{h_\sigma}, \\ \|(\mathcal{I} - \mathcal{I}_{h_u})(\varphi_{h_u}^i v_h)\|_{L^2(\Omega)} &\leq K \frac{h_u}{H} \|v_h\|_{L^2(\Omega)}, \quad \forall v_h \in \mathcal{V}_{h_u}. \end{aligned} \tag{8}$$

Proof Using Lemma 2, we know that

$$\|(\mathcal{I} - \mathcal{I}_{h_\sigma})(\varphi_{h_\sigma}^i \omega_h)\|_{[L^2(\Omega)]^d} \leq Kh_\sigma \|\varphi_{h_\sigma}^i\|_{W^{1,\infty}(\Omega)} \|\omega_h\|_{[L^2(\Omega)]^d} \leq K \frac{h_\sigma}{H} \|\omega_h\|_{[L^2(\Omega)]^d}.$$

This is the first inequality of (8).

In addition, by using the technique of Theorem 3.1 in Yang (2001), we can easily obtain

$$\begin{aligned} \|(\mathcal{I} - \mathcal{I}_{h_u})(\varphi_{h_u}^i v_h)\|_{L^2(\Omega)} &\leq Kh_u \min(\|\varphi_{h_u}^i\|_{W^{1,\infty}(\Omega)} \|v_h\|_{L^2(\Omega)}, \|\varphi_{h_u}^i\|_{H^1(\Omega)} \|v_h\|_{L^\infty(\Omega)}) \\ &\leq Kh_u \|\varphi_{h_u}^i\|_{W^{1,\infty}(\Omega)} \|v_h\|_{L^2(\Omega)} \leq K \frac{h_u}{H} \|v_h\|_{L^2(\Omega)}. \end{aligned}$$

That is the second inequality of (8). The proof of Lemma 3 is completed. □

Lemma 4 The following estimate

$$\begin{aligned} \left| a_n \left((\psi, w), (\omega, v) \right) - \sum_{i=1}^N a_n \left((\psi, w), (\mathcal{I}_{h_\sigma}(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega), \mathcal{I}_{h_u}(\varphi_{h_u}^i Q_{h_u}^i v)) \right) \right| \\ \leq K \left(\frac{h}{H} + \frac{\sqrt{\tau}}{H} \right) \|(\psi, w)\|_{a_n} \|(\omega_h, v_h)\|_{a_n} \end{aligned} \tag{9}$$

holds for each (ψ, w) and (ω, v) in $\mathcal{W}_{h_\sigma} \times \mathcal{V}_{h_u}$.

Proof It is easily seen that

$$\begin{aligned} &a_n \left((\psi, w), (\mathcal{I}_{h_\sigma}(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega), \mathcal{I}_{h_u}(\varphi_{h_u}^i Q_{h_u}^i v)) \right) \\ &= a_n \left((\psi, w), (\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega, \varphi_{h_u}^i Q_{h_u}^i v) \right) \\ &\quad + a_n \left((\psi, w), ((\mathcal{I}_{h_\sigma} - \mathcal{I})(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega), (\mathcal{I}_{h_u} - \mathcal{I})(\varphi_{h_u}^i Q_{h_u}^i v)) \right), \end{aligned}$$

and

$$\begin{aligned}
 & a_n \left((\psi, w), (\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega, \varphi_{h_u}^i Q_{h_u}^i v) \right) \\
 &= a_n \left((P_{h_\sigma}^i (\varphi_{h_\sigma}^i \psi), Q_{h_u}^i (\varphi_{h_u}^i w)), (\omega, v) \right) \\
 & \quad + \tau_n \left[\left(\frac{1}{c} (c w + \tau_n (\nabla \cdot \psi + q^n w)), (P_{h_\sigma}^i \omega) \nabla \varphi_{h_\sigma}^i \right) \right. \\
 & \quad - \left. \left(\frac{1}{c} \psi \nabla \varphi_{h_\sigma}^i, c Q_{h_u}^i v + \tau_n (\nabla \cdot (P_{h_\sigma}^i \omega) + q^n Q_{h_u}^i v) \right) \right. \\
 & \quad + \left. (\tilde{A}(\psi + A \nabla w + \mathbf{b}^n w), A \nabla \varphi_{h_u}^i Q_{h_u}^i v) \right. \\
 & \quad \left. - (\nabla \varphi_{h_u}^i w, P_{h_\sigma}^i \omega + A \nabla (Q_{h_u}^i v) + \mathbf{b}^n Q_{h_u}^i v) \right]
 \end{aligned}$$

and

$$a_n((\psi, w), (\omega, v)) = \sum_{i=1}^N a_n \left((\varphi_{h_\sigma}^i \psi, \varphi_{h_u}^i w), (\omega, v) \right).$$

Hence we have

$$\begin{aligned}
 & a_n((\psi, w), (\omega, v)) - \sum_{i=1}^N a_n((\psi, w), (\mathcal{I}_{h_\sigma}(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega), \mathcal{I}_{h_u}(\varphi_{h_u}^i Q_{h_u}^i v))) \\
 &= \sum_{i=1}^N a_n \left(((\mathcal{I} - P_{h_\sigma}^i)(\varphi_{h_\sigma}^i \psi), (\mathcal{I} - Q_{h_u}^i)(\varphi_{h_u}^i w)), (\omega, v) \right) \\
 & \quad - \sum_{i=1}^N a_n((\psi, w), ((\mathcal{I}_{h_\sigma} - \mathcal{I})(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega), (\mathcal{I}_{h_u} - \mathcal{I})(\varphi_{h_u}^i Q_{h_u}^i v))) \\
 & \quad - \tau_n \sum_{i=1}^N \left[\left(\frac{1}{c} (c w + \tau_n (\nabla \cdot \psi + q^n w)), (P_{h_\sigma}^i \omega) \nabla \varphi_{h_\sigma}^i \right) \right. \\
 & \quad - \left. \left(\frac{1}{c} \psi \nabla \varphi_{h_\sigma}^i, c Q_{h_u}^i v + \tau_n (\nabla \cdot (P_{h_\sigma}^i \omega) + q^n Q_{h_u}^i v) \right) \right. \\
 & \quad + \left. (\tilde{A}(\psi + A \nabla w + \mathbf{b}^n w), A \nabla \varphi_{h_u}^i Q_{h_u}^i v) \right. \\
 & \quad \left. - (\nabla \varphi_{h_u}^i w, P_{h_\sigma}^i \omega + A \nabla (Q_{h_u}^i v) + \mathbf{b}^n Q_{h_u}^i v) \right]. \tag{10}
 \end{aligned}$$

Noting that

$$\begin{aligned}
 & \|(\mathcal{I}_{h_\sigma} - \mathcal{I})(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega), (\mathcal{I}_{h_u} - \mathcal{I})(\varphi_{h_u}^i Q_{h_u}^i v)\|_{a_n} \\
 & \leq K \{ \|c(\mathcal{I}_{h_u} - \mathcal{I})(\varphi_{h_u}^i Q_{h_u}^i v)\|_{L^2(\Omega_i)} + \tau_n \|\nabla \cdot (\mathcal{I}_{h_\sigma} - \mathcal{I})(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega)\|_{L^2(\Omega_i)} \\
 & \quad + \tau_n \|q^n (\mathcal{I}_{h_u} - \mathcal{I})(\varphi_{h_u}^i Q_{h_u}^i v)\|_{L^2(\Omega_i)} + \sqrt{\tau_n} \left[\|(\mathcal{I}_{h_\sigma} - \mathcal{I})(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega)\|_{L^2(\Omega_i)} \right. \\
 & \quad \left. + \|A \nabla ((\mathcal{I}_{h_u} - \mathcal{I})(\varphi_{h_u}^i Q_{h_u}^i v))\|_{L^2(\Omega_i)} + \|\mathbf{b}^n (\mathcal{I}_{h_u} - \mathcal{I})(\varphi_{h_u}^i Q_{h_u}^i v)\|_{L^2(\Omega_i)} \right] \} \\
 & \leq K \left\{ (1 + \tau_n) \frac{h_u}{H} \|Q_{h_u}^i v\|_{L^2(\Omega_i)} \tau_n \frac{1}{h_\sigma} \frac{h_\sigma}{H} \|P_{h_\sigma}^i \omega\|_{[L^2(\Omega_i)]^d} \right. \\
 & \quad \left. + \sqrt{\tau_n} \left[\frac{h_\sigma}{H} \|P_{h_\sigma}^i \omega\|_{[L^2(\Omega_i)]^d} + \frac{1}{h_u} \frac{h_u}{H} \|Q_{h_u}^i v\|_{L^2(\Omega_i)} + \frac{h_u}{H} \|Q_{h_u}^i v\|_{L^2(\Omega_i)} \right] \right\} \\
 & \leq K \left(\frac{h}{H} + \frac{\sqrt{\tau}}{H} \right) \left\{ \|Q_{h_u}^i v\|_{L^2(\Omega_i)} + \sqrt{\tau_n} \left[\|P_{h_\sigma}^i \omega\|_{[L^2(\Omega_i)]^d} + \|Q_{h_u}^i v\|_{L^2(\Omega_i)} \right] \right\} \\
 & \leq K \left(\frac{h}{H} + \frac{\sqrt{\tau}}{H} \right) \| (P_{h_\sigma}^i \omega, Q_{h_u}^i v) \|_{a_n, \Omega_i},
 \end{aligned}$$

we have

$$\begin{aligned} & \left| \sum_{i=1}^N a_n((\psi, w), (\mathcal{I}_{h_\sigma} - \mathcal{I})(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega), (\mathcal{I}_{h_u} - \mathcal{I})(\varphi_{h_u}^i Q_{h_u}^i v)) \right| \\ & \leq K \left(\frac{h}{H} + \frac{\sqrt{\tau}}{H} \right) \|(\psi, w)\|_{a_n} \left[\sum_{i=1}^N \|(P_{h_\sigma}^i \omega, Q_{h_u}^i v)\|_{a_n, \Omega_i}^2 \right]^{1/2}, \\ & \left| \sum_{i=1}^N a_n \left((\mathcal{I} - P_{h_\sigma}^i)(\varphi_{h_\sigma}^i \psi), (\mathcal{I} - Q_{h_u}^i)(\varphi_{h_u}^i w), (\omega, v) \right) \right| \\ & = \left| \sum_{i=1}^N a_n \left((\mathcal{I} - \mathcal{I}_{h_\sigma})(\varphi_{h_\sigma}^i \psi), (\mathcal{I} - \mathcal{I}_{h_u})(\varphi_{h_u}^i w), (\mathcal{I} - P_{h_\sigma}^i)\omega, (\mathcal{I} - Q_{h_u}^i)v \right) \right| \\ & \leq K \left(\frac{h}{H} + \frac{\sqrt{\tau}}{H} \right) \|(\psi, w)\|_{a_n} \left[\sum_{i=1}^N \|(\mathcal{I} - P_{h_\sigma}^i)\omega, (\mathcal{I} - Q_{h_u}^i)v\|_{a_n, \Omega_i}^2 \right]^{1/2} \end{aligned}$$

and

$$\begin{aligned} & \tau_n \sum_{i=1}^N \left[\left| \left(\frac{1}{c}(cw + \tau_n(\nabla \cdot \psi + q^n w)), (P_{h_\sigma}^i \omega) \nabla \varphi_{h_\sigma}^i \right) \right| \right. \\ & \quad \left. + \left| \left(\frac{1}{c} \psi \nabla \varphi_{h_\sigma}^i, cQ_{h_u}^i v + \tau_n(\nabla \cdot (P_{h_\sigma}^i \omega) + q^n Q_{h_u}^i v) \right) \right| \right. \\ & \quad \left. + \left| (\tilde{A}(\psi + A \nabla w + \mathbf{b}^n w), A \nabla \varphi_{h_u}^i Q_{h_u}^i v) \right| \right. \\ & \quad \left. + \left| (\nabla \varphi_{h_u}^i w, P_{h_\sigma}^i \omega + A \nabla (Q_{h_u}^i v) + \mathbf{b}^n Q_{h_u}^i v) \right| \right] \\ & \leq K \frac{\sqrt{\tau_n}}{H} \left[\sum_{i=1}^N \|(\psi, w)\|_{a_n} \|(P_{h_\sigma}^i \omega, Q_{h_u}^i v)\|_{a_n, \Omega_i}^2 \right]^{1/2}. \end{aligned}$$

Substituting these estimates into (10) leads to (9). This ends the proof of Lemma 4. \square

For parallel algorithm, we have the following convergence result:

Theorem 1 *Let (σ, u) and (σ_h^n, u_h^n) are the solutions of the system (1) and parallel algorithm, respectively. If $h^{2m} = O(\tau)$, then there holds the following a priori error estimate*

$$\begin{aligned} & \max_n \|u^n - u_h^n\|_{L^2(\Omega)} + \max_n \|\sigma^n - \sigma_h^n\|_{[L^2(\Omega)]^d} \\ & \leq K \left\{ \left(\frac{h^2}{H^2} + \frac{\tau}{H^2} \right)^{\frac{m}{2}} + h_u^{k+1} + h_\sigma^r + \tau \right\}, \end{aligned} \tag{11}$$

where $h = \max(h_\sigma, h_u)$.

Proof of Theorem 1

It is easily seen that parallel algorithm is also equivalent to use an iteration with initial values $(\sigma_h^{n-1}, u_h^{n-1})$ to solve the following equation: $(\hat{\sigma}_h^n, \hat{u}_h^n) \in \mathcal{W}_{h_\sigma} \times \mathcal{V}_{h_u}$ such that for any $(\omega_h, v_h) \in \mathcal{W}_{h_\sigma} \times \mathcal{V}_{h_u}$

$$\begin{aligned}
 a_n((\hat{\sigma}_h^n, \hat{u}_h^n), (\omega_h, v_h)) &= \left(\frac{1}{c}(cu_h^{n-1} + \tau_n f^n), cv_h + \tau_n(\nabla \cdot \omega_h + q^n v_h) \right) \\
 &\quad + \tau_n \left(\tilde{A}(\sigma_h^{n-1} + A\nabla u_h^{n-1} + \mathbf{b}^{n-1} u_h^{n-1}), \omega_h + A\nabla v_h + \mathbf{b}^n v_h \right).
 \end{aligned} \tag{12}$$

From (12) we have

$$\begin{aligned}
 a_n((\sigma_h^n, u_h^n), (\omega_h, v_h)) &= \left(\frac{1}{c}(cu_h^{n-1} + \tau_n f^n), cv_h + \tau_n(\nabla \cdot \omega_h + q^n v_h) \right) \\
 &\quad + \tau_n \left(\tilde{A}(\sigma_h^{n-1} + A\nabla u_h^{n-1} + \mathbf{b}^{n-1} u_h^{n-1}), \omega_h + A\nabla v_h + \mathbf{b}^n v_h \right) \\
 &\quad + a_n((\sigma_h^n - \hat{\sigma}_h^n, u_h^n - \hat{u}_h^n), (\omega_h, v_h)).
 \end{aligned} \tag{13}$$

Let $\theta^n = u_h^n - w_h^n$, $\rho^n = w_h^n - u^n$, $\pi^n = \sigma_h^n - \varrho_h^n$ and $\eta^n = \varrho_h^n - \sigma^n$. Subtracting (4) from (13), we can get

$$\begin{aligned}
 a_n((\pi^n, \theta^n), (\omega_h, v_h)) &= \left(\theta^{n-1}, cv_h + \tau_n(\nabla \cdot \omega_h + q^n v_h) \right) \\
 &\quad + \tau_n \left(\tilde{A}(\pi^{n-1} + A\nabla \theta^{n-1} + \mathbf{b}^{n-1} \theta^{n-1}), \omega_h + A\nabla v_h + \mathbf{b}^n v_h \right) \\
 &\quad + a_n((\sigma_h^n - \hat{\sigma}_h^n, u_h^n - \hat{u}_h^n), (\omega_h, v_h)).
 \end{aligned} \tag{14}$$

Lemma 5 For parallel algorithm, we have the estimate

$$\|(\sigma_h^n - \hat{\sigma}_h^n, u_h^n - \hat{u}_h^n)\|_{a_n} \leq K \left(\frac{h^2}{H^2} + \frac{\tau}{H^2} \right)^{\frac{m}{2}} \|(\sigma_h^{n-1} - \hat{\sigma}_h^n, u_h^{n-1} - \hat{u}_h^n)\|_{a_n}. \tag{15}$$

Proof From (5), we have

$$\begin{aligned}
 a_n((\varepsilon_j^i, e_j^i), (\omega_h, v_h)) &= a_n((\varepsilon_j^i, e_j^i), (P_{h_\sigma}^i \omega_h, Q_{h_u}^i v_h)) \\
 &= a_n((\hat{\sigma}^n - \tilde{\sigma}_{j-1}^n, \hat{u}^n - \tilde{u}_{j-1}^n), (\mathcal{I}_{h_\sigma}(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega_h), \mathcal{I}_{h_u}(\varphi_{h_u}^i Q_{h_u}^i v_h))).
 \end{aligned} \tag{16}$$

In addition, from parallel algorithm we can obtain the following equation

$$\begin{aligned}
 &a_n((\tilde{\sigma}_j^n - \hat{\sigma}_h^n, \tilde{u}_j^n - \hat{u}_h^n), (\omega_h, v_h)) \\
 &= a_n((\tilde{\sigma}_{j-1}^n - \hat{\sigma}_h^n, \tilde{u}_{j-1}^n - \hat{u}_h^n), (\omega_h, v_h)) + a_n \left(\left(\sum_{i=1}^N \varepsilon_j^i, \sum_{i=1}^N e_j^i \right), (\omega_h, v_h) \right) \\
 &= a_n((\tilde{\sigma}_{j-1}^n - \hat{\sigma}_h^n, \tilde{u}_{j-1}^n - \hat{u}_h^n), (\omega_h, v_h)) \\
 &\quad + a_n((\hat{\sigma}_h^n - \tilde{\sigma}_{j-1}^n, \hat{u}_h^n - \tilde{u}_{j-1}^n), \left(\sum_{i=1}^N \mathcal{I}_{h_\sigma}(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega_h), \sum_{i=1}^N \mathcal{I}_{h_u}(\varphi_{h_u}^i Q_{h_u}^i v_h) \right)).
 \end{aligned} \tag{17}$$

Taking $(\omega_h, v_h) = (\tilde{\sigma}_j^n - \hat{\sigma}_h^n, \tilde{u}_j^n - \hat{u}_h^n)$ in (17) and using Lemma 4, we have

$$\|(\tilde{\sigma}_j^n - \hat{\sigma}_h^n, \tilde{u}_j^n - \hat{u}_h^n)\|_{a_n}^2 \leq K \left(\frac{h^2}{H^2} + \frac{\tau}{H^2} \right) \|(\tilde{\sigma}_{j-1}^n - \hat{\sigma}_h^n, \tilde{u}_{j-1}^n - \hat{u}_h^n)\|_{a_n}^2. \tag{18}$$

Thus, we have

$$\|(\tilde{\sigma}_m^n - \hat{\sigma}_h^n, \tilde{u}_m^n - \hat{u}_h^n)\|_{a_n}^2 \leq K \left(\frac{h^2}{H^2} + \frac{\tau}{H^2} \right)^m \|(\tilde{\sigma}_0^n - \hat{\sigma}_h^n, \tilde{u}_0^n - \hat{u}_h^n)\|_{a_n}^2. \tag{19}$$

That is the inequality (15). This ends the proof of Lemma 5. □

Hence, we need to estimate the bounds of $\sigma_h^{n-1} - \hat{\sigma}_h^n$ and $u_h^{n-1} - \hat{u}_h^n$.

Lemma 6 For parallel algorithm, we have the following estimate

$$\begin{aligned} & \|(\hat{\sigma}_h^n - \sigma_h^{n-1}, \hat{u}_h^n - u_h^{n-1})\|_{a_n}^2 \\ & \leq K \tau_n \left\{ \int_{t^{n-1}}^{t^n} \left\| \left(\frac{\partial \varrho_h}{\partial t}, \frac{\partial w_h}{\partial t} \right) \right\|_{a_n}^2 dt + \tau_n [\|\nabla \cdot \pi^{n-1}\|_{L^2(\Omega)}^2 + \|\theta^{n-1}\|_{L^2(\Omega)}^2] \right\}. \end{aligned} \tag{20}$$

Proof From (14) we have

$$\begin{aligned} & a_n((\hat{\sigma}_h^n - \sigma_h^{n-1}, \hat{u}_h^n - u_h^{n-1}), (\omega_h, v_h)) \\ & = a_n((\varrho_h^n - \varrho_h^{n-1}, w_h^n - w_h^{n-1}), (\omega_h, v_h)) \\ & \quad - \tau_n \left(\frac{1}{c} (\nabla \cdot \pi^{n-1} + q^n \theta^{n-1}), c v_h + \tau_n (\nabla \cdot \omega_h + q^n v_h) \right) \\ & \quad - \tau_n (\tilde{A}(\mathbf{b}^n - \mathbf{b}^{n-1}) \theta^{n-1}, \omega_h + A \nabla v_h + \mathbf{b}^n v_h) \end{aligned} \tag{21}$$

Taking $(\omega_h, v_h) = (\hat{\sigma}_h^n - \sigma_h^{n-1}, \hat{u}_h^n - u_h^{n-1})$ in (21) and using the inequality $ab \leq \frac{1}{\delta} a^2 + \delta b^2$, we can obtain

$$\begin{aligned} & \|(\hat{\sigma}_h^n - \sigma_h^{n-1}, \hat{u}_h^n - u_h^{n-1})\|_{a_n}^2 \\ & \leq K \tau_n \left\{ \int_{t^{n-1}}^{t^n} \left\| \left(\frac{\partial \varrho_h}{\partial t}, \frac{\partial w_h}{\partial t} \right) \right\|_{a_n}^2 dt + \tau_n [\|\nabla \cdot \pi^{n-1}\|_{L^2(\Omega)}^2 + \|\theta^{n-1}\|_{L^2(\Omega)}^2] \right\} \\ & \quad + \delta \|(\hat{\sigma}_h^n - \sigma_h^{n-1}, \hat{u}_h^n - u_h^{n-1})\|_{a_n}^2 \end{aligned}$$

Hence, when we choose sufficiently small δ , we can obtain the estimate (20). This ends the proof of Lemma 6. □

Finally, we prove Theorem 1.

Proof Let $(\omega_h, v_h) = (\pi^n, \theta^n - \theta^{n-1})$ in (14), we have

$$\begin{aligned} & a_n((\pi^n, \theta^n - \theta^{n-1}), (\pi^n, \theta^n - \theta^{n-1})) \\ & = \tau_n \left(\tilde{A}(\pi^{n-1} - \pi^n), \pi^n + A \nabla(\theta^n - \theta^{n-1}) + \mathbf{b}^n(\theta^n - \theta^{n-1}) \right) \\ & \quad - \tau_n \left(\frac{1}{c} (\nabla \cdot \pi^n + q^n \theta^{n-1}), c(\theta^n - \theta^{n-1}) + \tau_n (\nabla \cdot \pi^n + q^n(\theta^n - \theta^{n-1})) \right) \\ & \quad + \tau_n (\tilde{A}(\mathbf{b}^{n-1} - \mathbf{b}^n) \theta^{n-1}, \pi^n + A \nabla(\theta^n - \theta^{n-1}) + \mathbf{b}^n(\theta^n - \theta^{n-1})) \\ & \quad + a_n((\sigma_h^n - \hat{\sigma}_h^n, u_h^n - \hat{u}_h^n), (\pi^n, \theta^n - \theta^{n-1})). \end{aligned}$$

Since

$$\begin{aligned}
 & a_n((\pi^n, \theta^n - \theta^{n-1}), (\pi^n, \theta^n - \theta^{n-1})) - \tau_n(\tilde{A}(\pi^{n-1} - \pi^n), \pi^n) \\
 &= (c(\theta^n - \theta^{n-1}), \theta^n - \theta^{n-1}) + \tau_n \left[(A\nabla(\theta^n - \theta^{n-1}), \nabla(\theta^n - \theta^{n-1})) \right. \\
 &\quad + (\tilde{A}\mathbf{b}^n(\theta^n - \theta^{n-1}), \theta^n - \theta^{n-1}) + \tau_n \left(\frac{1}{c} \nabla \cdot \pi^n, \nabla \cdot \pi^n \right) \\
 &\quad \left. + \tau_n \left(\frac{1}{c} q^n(\theta^n - \theta^{n-1}), q^n(\theta^n - \theta^{n-1}) \right) \right] + \tau_n(\tilde{A}\pi^n, \pi^n) \\
 &\quad + \frac{\tau_n}{2} [(\tilde{A}\pi^n, \pi^n) - (\tilde{A}\pi^{n-1}, \pi^{n-1}) + (\tilde{A}(\pi^n - \pi^{n-1}), \pi^n - \pi^{n-1})] \\
 &\quad + 2\tau_n \left(\frac{1}{c} (c(\theta^n - \theta^{n-1}) + \tau_n \nabla \cdot \pi^n), q^n(\theta^n - \theta^{n-1}) \right) \\
 &\quad + 2\tau_n(\tilde{A} + \nabla(\theta^n - \theta^{n-1}), \mathbf{b}^n(\theta^n - \theta^{n-1})),
 \end{aligned}$$

we have

$$\begin{aligned}
 & (c(\theta^n - \theta^{n-1}), \theta^n - \theta^{n-1}) + \tau_n \left[(A\nabla(\theta^n - \theta^{n-1}), \nabla(\theta^n - \theta^{n-1})) \right. \\
 &\quad + (\tilde{A}\mathbf{b}^n(\theta^n - \theta^{n-1}), \theta^n - \theta^{n-1}) + \tau_n \left(\frac{1}{c} \nabla \cdot \pi^n, \nabla \cdot \pi^n \right) \\
 &\quad \left. + \tau_n \left(\frac{1}{c} q^n(\theta^n - \theta^{n-1}), q^n(\theta^n - \theta^{n-1}) \right) \right] + \tau_n(\tilde{A}\pi^n, \pi^n) \\
 &\quad + \frac{\tau_n}{2} [(\tilde{A}\pi^n, \pi^n) + (\tilde{A}(\pi^n - \pi^{n-1}), \pi^n - \pi^{n-1})] \\
 &= \frac{\tau_n}{2} (\tilde{A}\pi^{n-1}, \pi^{n-1}) + \tau_n \left(\tilde{A}(\pi^{n-1} - \pi^n), A\nabla(\theta^n - \theta^{n-1}) + \mathbf{b}^n(\theta^n - \theta^{n-1}) \right) \\
 &\quad - \tau_n \left(\frac{1}{c} (\nabla \cdot \pi^n + q^n \theta^{n-1}), c(\theta^n - \theta^{n-1}) + \tau_n (\nabla \cdot \pi^n + q^n(\theta^n - \theta^{n-1})) \right) \\
 &\quad + \tau_n \left(\tilde{A}(\mathbf{b}^{n-1} - \mathbf{b}^n) \theta^{n-1}, \pi^n + A\nabla(\theta^n - \theta^{n-1}) + \mathbf{b}^n(\theta^n - \theta^{n-1}) \right) \\
 &\quad + a_n((\sigma_h^n - \hat{\sigma}_h^n, u_h^n - \hat{u}_h^n), (\pi^n, \theta^n - \theta^{n-1})). \tag{22}
 \end{aligned}$$

Next, we estimate the terms on the right-hand side of the error equation (22).

It is clear that

$$\begin{aligned}
 & \tau_n(\tilde{A}(\pi^{n-1} - \pi^n), A\nabla(\theta^n - \theta^{n-1})) \\
 & \leq \frac{\tau_n}{2} \left[(\tilde{A}(\pi^n - \pi^{n-1}), \pi^n - \pi^{n-1}) + (A\nabla(\theta^n - \theta^{n-1}), \nabla(\theta^n - \theta^{n-1})) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 & \tau_n \left| \left(\tilde{A}(\pi^{n-1} - \pi^n), \mathbf{b}^n(\theta^n - \theta^{n-1}) \right) \right| + \tau_n \left| \left(\frac{1}{c}(\nabla \cdot \pi^n + q^n \theta^{n-1}), c(\theta^n - \theta^{n-1}) \right) \right. \\
 & \quad \left. + \tau_n(\nabla \cdot \pi^n + q^n(\theta^n - \theta^{n-1})) \right| \\
 & + \tau_n \left| \left(\tilde{A}(\mathbf{b}^{n-1} - \mathbf{b}^n)\theta^{n-1}, \pi^n + A\nabla(\theta^n - \theta^{n-1}) + \mathbf{b}^n(\theta^n - \theta^{n-1}) \right) \right| \\
 & + |a_n((\sigma_h^n - \hat{\sigma}_h^n, u_h^n - \hat{u}_h^n), (\pi^n, \theta^n - \theta^{n-1}))| \\
 & \leq K \tau_n \left\{ \left(\frac{h^2}{H^2} + \frac{\tau}{H^2} \right)^m \int_{t^{n-1}}^{t^n} \left\| \left(\frac{\partial \sigma_h}{\partial t}, \frac{\partial w_h}{\partial t} \right) \right\|_{a_n}^2 dt + \tau_n [\|\theta^{n-1}\|_{L^2(\Omega)}^2 \right. \\
 & \quad \left. + \|\nabla(\theta^n - \theta^{n-1})\|_{[L^2(\Omega)]^d}^2 + \|\pi^{n-1}\|_{[L^2(\Omega)]^d}^2 + \|\pi^n\|_{[L^2(\Omega)]^d}^2 \right. \\
 & \quad \left. + \|\nabla \cdot \pi^{n-1}\|_{L^2(\Omega)}^2 \right\} + \delta [\|\theta^n - \theta^{n-1}\|_{L^2(\Omega)}^2 + \tau_n^2 \|\nabla \cdot \pi^n\|_{L^2(\Omega)}^2].
 \end{aligned}$$

Substituting the above estimates into (22) and then summing it up from 1 to n , we get

$$\begin{aligned}
 & \|\pi^n\|_{[L^2(\Omega)]^d}^2 + \sum_{j=1}^n \tau_j \left[\|\bar{\partial}_t \theta^j\|_{L^2(\Omega)}^2 + \tau_j \|\nabla \bar{\partial}_t \theta^j\|_{[L^2(\Omega)]^d}^2 + \tau_j \|\nabla \cdot \pi^j\|_{L^2(\Omega)}^2 \right] \\
 & \leq K \left\{ \left(\frac{h^2}{H^2} + \frac{\tau}{H^2} \right)^m \int_0^{t^n} \left\| \left(\frac{\partial \sigma_h}{\partial t}, \frac{\partial w_h}{\partial t} \right) \right\|_{a_n}^2 dt + \sum_{j=1}^n \tau_j \left[\|\pi^j\|_{[L^2(\Omega)]^d}^2 \right. \right. \\
 & \quad \left. \left. + \|\theta^j\|_{L^2(\Omega)}^2 + \tau_j^2 \|\nabla \bar{\partial}_t \theta^j\|_{[L^2(\Omega)]^d}^2 \right] \right\}. \tag{23}
 \end{aligned}$$

Applying a known inequality

$$\|\theta^n\|_{L^2(\Omega)}^2 \leq \|\theta^0\|_{L^2(\Omega)}^2 + \delta \sum_{j=1}^n \tau_j \|\bar{\partial}_t \theta^j\|_{L^2(\Omega)}^2 + K \sum_{j=1}^n \tau_j \|\theta^j\|_{L^2(\Omega)}^2$$

and discrete Gronwall’s lemma to (23), we derive that

$$\max_n \|\theta^n\|_{L^2(\Omega)} + \max_n \|\pi^n\|_{[L^2(\Omega)]^d} \leq K \left(\frac{h^2}{H^2} + \frac{\tau}{H^2} \right)^{\frac{m}{2}}. \tag{24}$$

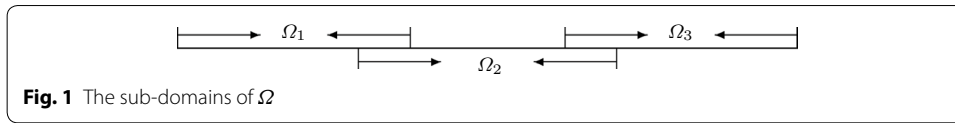
Using Lemma 1, we can obtain the estimate (11). The proof of Theorem 1 is complete. \square

Numerical results

As in Zhang and Yang (2011), we first consider the one dimensional convection–diffusion problem:

$$\frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} - b \frac{\partial u}{\partial x} + u = f \quad x \in [0, 1], \quad 0 \leq t \leq T. \tag{25}$$

We divide the domain $[0, 1]$ into three sub-domains: $\Omega_1 = \left[0, \frac{1}{3} + \frac{H}{2}\right]$, $\Omega_2 = \left[\frac{1}{3} - \frac{H}{2}, \frac{2}{3} + \frac{H}{2}\right]$ and $\Omega_3 = \left[\frac{2}{3} - \frac{H}{2}, 1\right]$, where H is the overlapping degree (see Fig. 1).



We use piecewise linear polynomial spaces, set $h_u = h_\sigma = h$ and take the linear unit decomposition functions as in Zhang et al. (2011). We define the L^2 -norm error as follows:

$$\|(e, E)\|_2^2 = \max_n \|u^n - u_h^n\|_{L^2(\Omega)} + \max_n \|\sigma^n - \sigma_h^n\|_{L^2(\Omega)},$$

and the L^∞ -norm error

$$\|(e, E)\|_\infty = \max_n (|e^n|, |E^n|).$$

Experiment I In this experiment, the exact solution is chosen as $u = e^t \sin^2 \pi x$. Set $T = 1$, and $b = 1$. For different parameters a, h, τ and the iterative number m at each time step, we give L^2 -norm errors and the L^∞ -norm errors in Tables 1, 2 and 3. These numerical results suggest that we can get a good result for convection–diffusion problem using parallel algorithm, even iterating only one or two cycle at each time step. Moreover, these numerical results also imply that the errors caused by decomposing domain decrease as the discretization parameters h and τ decrease and increase as the overlapping degree H becomes small, which are coincided with our theoretical result.

Table 1 $H = \frac{1}{6}, h = \tau$

h	m	$a = 1$		$a = 1e-2$		$a = 1e-4$	
		$\ \cdot\ _2$	$\ \cdot\ _\infty$	$\ \cdot\ _2$	$\ \cdot\ _\infty$	$\ \cdot\ _2$	$\ \cdot\ _\infty$
$\frac{1}{48}$	*	2.9866e-2	1.9250e-1	8.6118e-3	2.6493e-2	7.4344e-3	1.5421e-2
$\frac{1}{48}$	1	4.0090e-2	2.0819e-1	8.5534e-3	2.5531e-2	7.4495e-3	1.4833e-2
$\frac{1}{48}$	2	4.0086e-2	1.7485e-1	8.6384e-3	2.5532e-2	7.4785e-3	1.5375e-2
$\frac{1}{48}$	3	4.1043e-2	1.7967e-1	8.6403e-3	2.5532e-2	7.4790e-3	1.5379e-2
$\frac{1}{48}$	4	4.1164e-2	1.7947e-1	8.6403e-3	2.5532e-2	7.4790e-3	1.5379e-2
$\frac{1}{96}$	*	1.4800e-2	9.5488e-2	4.9541e-3	2.5220e-2	3.6348e-3	7.4154e-3
$\frac{1}{96}$	1	1.7513e-2	1.0523e-1	4.9351e-3	2.4967e-2	3.6397e-3	7.3584e-3
$\frac{1}{96}$	2	1.7951e-2	9.1683e-2	4.9457e-3	2.4967e-2	3.6420e-3	7.4145e-3
$\frac{1}{96}$	3	1.8186e-2	9.3327e-2	4.9458e-3	2.4967e-2	3.6420e-3	7.4147e-3
$\frac{1}{96}$	4	1.8219e-2	9.3540e-2	4.9458e-3	2.4967e-2	3.6420e-3	7.4147e-3
$\frac{1}{192}$	*	7.5407e-3	4.7396e-2	2.9959e-3	2.0611e-2	1.8231e-3	4.2963e-3
$\frac{1}{192}$	1	7.9423e-3	5.1561e-2	2.9899e-3	2.0539e-2	1.8239e-3	4.2920e-3
$\frac{1}{192}$	2	8.1633e-3	4.7424e-2	2.9912e-3	2.0539e-2	1.8240e-3	4.2920e-3
$\frac{1}{192}$	3	8.1986e-3	4.7096e-2	2.9912e-3	2.0539e-2	1.8240e-3	4.2920e-3
$\frac{1}{192}$	4	8.2016e-3	4.7458e-2	2.9912e-3	2.0539e-2	1.8240e-3	4.2920e-3

* The numerical results by least-squares algorithm

Table 2 $H = \frac{1}{12}, h = \tau$

h	m	$\alpha = 1$		$\alpha = 1e-2$		$\alpha = 1e-4$	
		$\ \cdot \ _2$	$\ \cdot \ _\infty$	$\ \cdot \ _2$	$\ \cdot \ _\infty$	$\ \cdot \ _2$	$\ \cdot \ _\infty$
$\frac{1}{48}$	*	2.9866e-2	1.9250e-1	8.6118e-3	2.6493e-2	7.4344e-3	1.5421e-2
$\frac{1}{48}$	1	8.4704e-2	2.2515e-1	8.7074e-3	2.4202e-2	7.5866e-3	1.6164e-2
$\frac{1}{48}$	2	8.0447e-2	1.9139e-1	8.7352e-3	2.4202e-2	7.5212e-3	1.5371e-2
$\frac{1}{48}$	3	7.9085e-2	1.8885e-1	8.7459e-3	2.4202e-2	7.5203e-3	1.5388e-2
$\frac{1}{48}$	4	7.7713e-2	1.8467e-1	8.7464e-3	2.4202e-2	7.5202e-3	1.5389e-2
$\frac{1}{96}$	*	1.4800e-2	9.5488e-2	4.9541e-3	2.5220e-2	3.6348e-3	7.4154e-3
$\frac{1}{96}$	1	3.7063e-2	1.0595e-1	4.9336e-3	2.4609e-2	3.6506e-3	7.4109e-3
$\frac{1}{96}$	2	3.5576e-2	8.4116e-2	4.9480e-3	2.4609e-2	3.6484e-3	7.4152e-3
$\frac{1}{96}$	3	3.5206e-2	8.2344e-2	4.9480e-3	2.4609e-2	3.6481e-3	7.4153e-3
$\frac{1}{96}$	4	3.4930e-2	8.0713e-2	4.9480e-3	2.4609e-2	3.6481e-3	7.4153e-3
$\frac{1}{192}$	*	7.5407e-3	4.7396e-2	2.9959e-3	2.0611e-2	1.8231e-3	4.2963e-3
$\frac{1}{192}$	1	1.3608e-2	5.2234e-2	2.9850e-3	2.0431e-2	1.8251e-3	4.2847e-3
$\frac{1}{192}$	2	1.3257e-2	4.1956e-2	2.9875e-3	2.0431e-2	1.8249e-3	4.2847e-3
$\frac{1}{192}$	3	1.3210e-2	4.2262e-2	2.9875e-3	2.0431e-2	1.8249e-3	4.2847e-3
$\frac{1}{192}$	4	1.3183e-2	4.2067e-2	2.9875e-3	2.0431e-2	1.8249e-3	4.2847e-3

* The numerical results by least-squares algorithm

Table 3 $H = \frac{1}{24}, h = \tau$

h	m	$\alpha = 1$		$\alpha = 1e-2$		$\alpha = 1e-4$	
		$\ \cdot \ _2$	$\ \cdot \ _\infty$	$\ \cdot \ _2$	$\ \cdot \ _\infty$	$\ \cdot \ _2$	$\ \cdot \ _\infty$
$\frac{1}{48}$	*	2.9866e-2	1.9250e-1	8.6118e-3	2.6493e-2	7.4344e-3	1.5421e-2
$\frac{1}{48}$	1	2.0573e-1	4.6896e-1	9.1964e-3	2.8774e-2	8.0709e-3	2.1458e-2
$\frac{1}{48}$	2	2.0179e-1	4.5896e-1	8.9982e-3	2.1092e-2	7.6579e-3	1.5340e-2
$\frac{1}{48}$	3	2.0061e-1	4.5878e-1	9.0836e-3	2.2172e-2	7.6797e-3	1.5379e-2
$\frac{1}{48}$	4	1.9834e-1	4.5297e-1	9.1164e-3	2.2495e-2	7.6884e-3	1.5387e-2
$\frac{1}{96}$	*	1.4800e-2	9.5488e-2	4.9541e-3	2.5220e-2	3.6348e-3	7.4154e-3
$\frac{1}{96}$	1	1.1518e-1	2.8223e-1	4.8606e-3	2.3682e-2	3.6679e-3	7.6455e-3
$\frac{1}{96}$	2	1.1342e-1	2.7485e-1	4.8758e-3	2.3682e-2	3.6564e-3	7.4153e-3
$\frac{1}{96}$	3	1.1266e-1	2.7333	4.8861e-3	2.3682e-2	3.6596e-3	7.4153e-3
$\frac{1}{96}$	4	1.1186e-1	2.6995e-1	4.8874e-3	2.3682e-2	3.6598e-3	7.4153e-3
$\frac{1}{192}$	*	7.5407e-3	4.7396e-2	2.9959e-3	2.0611e-2	1.8231e-3	4.2963e-3
$\frac{1}{192}$	1	5.5060e-2	1.4802e-1	2.9586e-3	2.0194e-2	1.8251e-3	4.2712e-3
$\frac{1}{192}$	2	5.4280e-2	1.4441e-1	2.9635e-3	2.0194e-2	1.8259e-3	4.2712e-3
$\frac{1}{192}$	3	5.3930e-2	1.4334e-1	2.9641e-3	2.0194e-2	1.8261e-3	4.2712e-3
$\frac{1}{192}$	4	5.3683e-2	1.4217e-1	2.9641e-3	2.0194e-2	1.8261e-3	4.2712e-3

* The numerical results by least-squares algorithm

Experiment II As in Zhang and Yang (2011), we select the right-hand side function with complex structure and the initial condition as follows:

$$\begin{cases} f(x, t) = 100e^{t-\frac{x}{5}} \cos(8\pi xt) \sin^2(7\pi x), \\ u^0(x) = 0. \end{cases}$$

Choosing $H = 1/12$, $h = \tau = 1/48$, $b = 1$, and $a = 1e-4$, we observe numerical results at different time (see Figs. 2, 3). We use “*” to denote u_h and σ_h , the values of the parallel algorithm and use “-” to denote w_h and ϱ_h , the values of least-squares algorithm. These figures clearly show that u_h, σ_h approximate to w_h and ϱ_h at different time, respectively, which is coincided with our theoretical analysis.

Next, we consider the two dimensional convection–diffusion problem:

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla \cdot \sigma + u = f, & x \in \Omega, 0 < t < T, \\ \sigma + \mathbf{A}\nabla u + \mathbf{b}u = 0, & x \in \Omega, 0 < t < T, \end{cases} \tag{26}$$

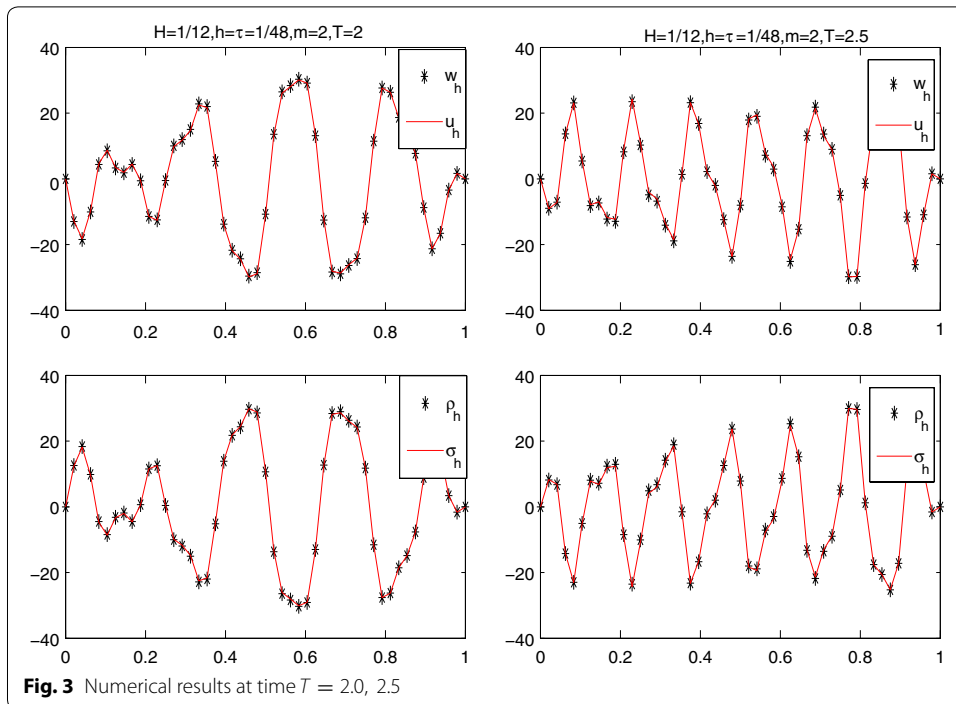
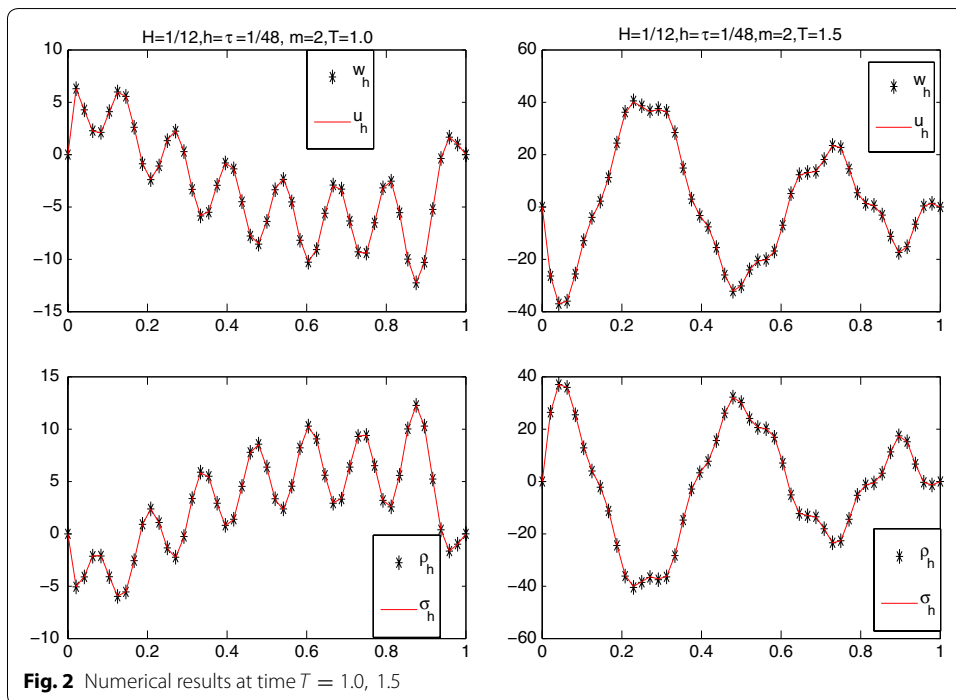
where $\Omega = [0, 1] \times [0, 1]$, $\mathbf{A} = a\mathbf{E}$, \mathbf{E} is the unit matrix, and $\mathbf{b} = (1, 1)^T$. We divide Ω into four sub-domains: $\Omega_1 = [0, 0.6] \times [0, 0.6]$, $\Omega_2 = [0.4, 1] \times [0, 0.6]$, $\Omega_3 = [0, 0.6] \times [0.4, 1]$, $\Omega_4 = [0.4, 1] \times [0.4, 1]$, see Fig. 4.

In this section, we use piecewise linear polynomial spaces. And We take the linear unit decomposition functions $\{\varphi_i\}_{i=1}^4$ as follows:

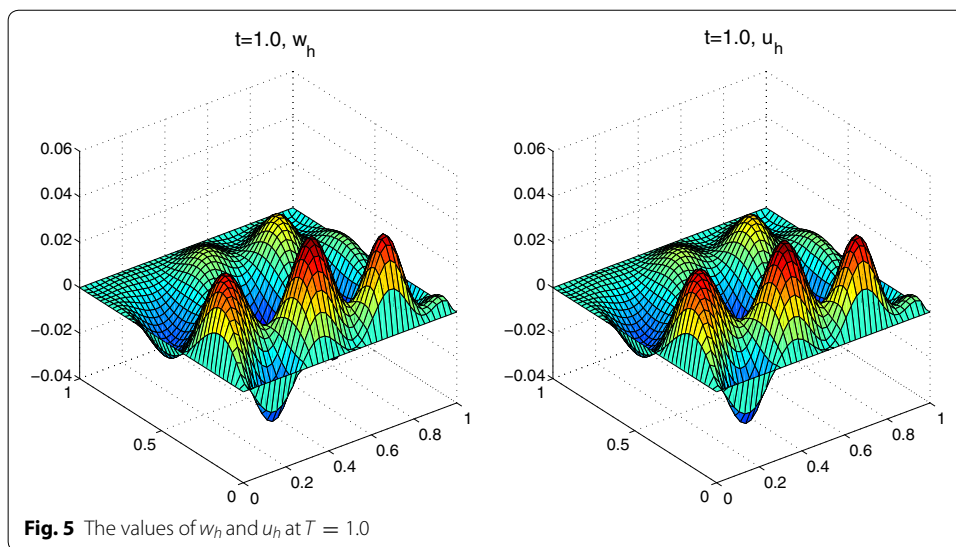
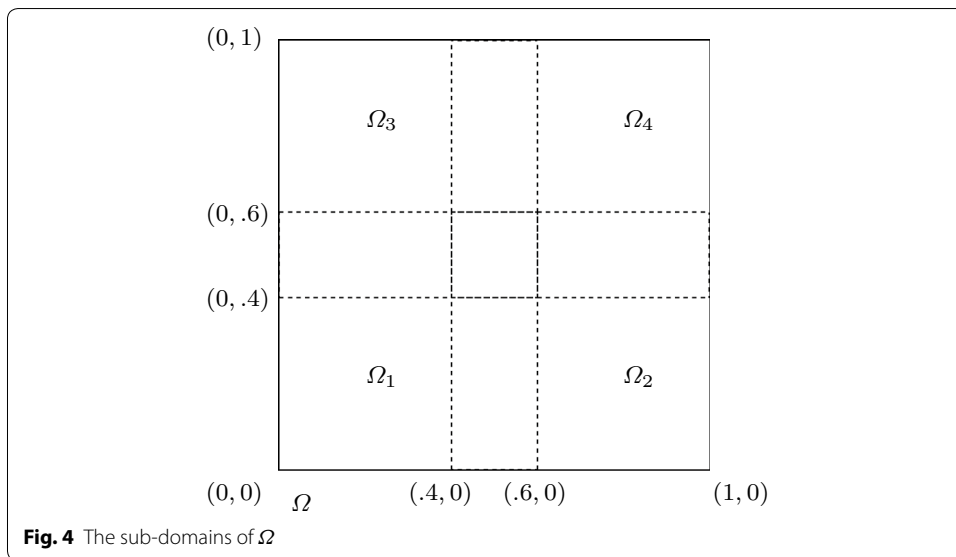
$$\begin{aligned} \varphi_1(x, y) &= \begin{cases} 1, & (x, y) \in [0, 0.4] \times [0, 0.4], \\ 3 - 5y, & (x, y) \in [0, 0.4] \times [0.4, 0.6], \\ 3 - 5x, & (x, y) \in [0.4, 0.6] \times [0, 0.4], \\ \frac{3}{2} - \frac{5}{4}(x + y), & \text{otherwise,} \end{cases} \\ \varphi_2(x, y) &= \begin{cases} 1, & (x, y) \in [0.6, 1] \times [0, 0.4], \\ 3 - 5y, & (x, y) \in [0.6, 1] \times [0.4, 0.6], \\ 5x - 2, & (x, y) \in [0.4, 0.6] \times [0, 0.4], \\ \frac{1}{4} - \frac{5}{4}(x - y), & \text{otherwise,} \end{cases} \\ \varphi_3(x, y) &= \begin{cases} 1, & (x, y) \in [0, 0.4] \times [0, 0.4], \\ 5y - 2, & (x, y) \in [0, 0.4] \times [0.4, 0.6], \\ 3 - 5x, & (x, y) \in [0.4, 0.6] \times [0.6, 1], \\ \frac{1}{4} + \frac{5}{4}(y - x), & \text{otherwise,} \end{cases} \\ \varphi_4(x, y) &= \begin{cases} 1, & (x, y) \in [0.6, 1] \times [0.6, 1], \\ 5y - 2, & (x, y) \in [0.6, 1] \times [0.4, 0.6], \\ 5x - 2, & (x, y) \in [0.4, 0.6] \times [0.6, 1], \\ \frac{5}{4}(x + y) - 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Experiment III Here we still select the same right-hand side function with complex structure and the initial condition as in Zhang and Yang (2011),

$$\begin{cases} f(x, t) = e^{y^3-x^2-2t} \sin(3\pi x - 6y + t^2) \cos(4\pi yt), \\ u^0(x) = 0. \end{cases}$$

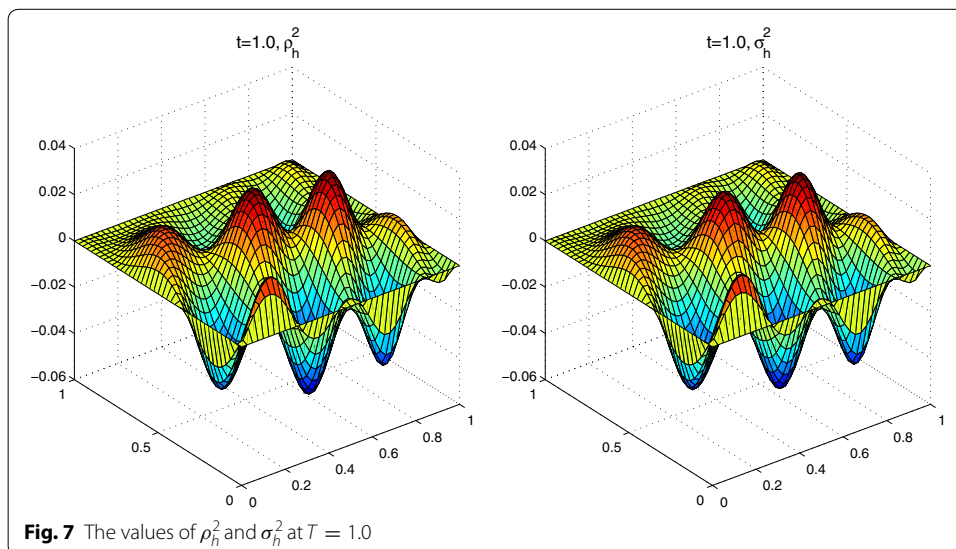
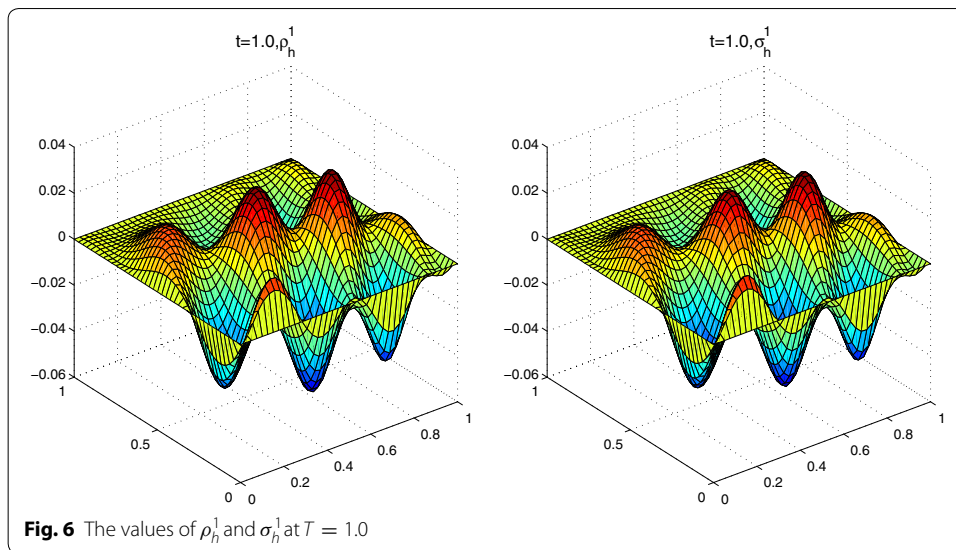


Set $H = 0.2$, $h = \tau = 1/40$, and $a = 1e-2$, $T = 1.0$, $m = 1$. We can get Figs. 5, 6 and 7. These results suggest that the values $u_h, \sigma_h = (\sigma_h^1, \sigma_h^2)$ by parallel algorithm approximate to w_h and the values $\rho_h = (\rho_h^1, \rho_h^2)$ by least-squares scheme respectively, which implies that our method is valid for two-dimensional problem.



Conclusions

In this paper, combined subspace correction method with least-squares mixed element procedure, a new class of parallel domain decomposition algorithm is proposed to solve convection–diffusion problem. The convergence of approximate solution, and the dependence of the convergent rate on the spacial mesh size, time increment, iteration number and sub-domains overlapping degree are studied. Both theoretical analysis and numerical experiments indicate the full parallelization of the algorithms and very good approximate property.



In fact, though we consider the convection–diffusion problem in this paper, we can extend our method to other complex problems, e.g. saltwater intrusion problem, aerodynamic problems, nuclear waste disposal, etc., which are our future work.

Authors' contributions

JSZ designed the algorithm and wrote the main manuscript text. HFF and HG did some the experiments and prepared all the figures. All the authors contributed to the discussion of the results. All authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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