Sun and Xi *SpringerPlus (2016)5:1469*
DOI 10.1186/s40064-016-3139-8

SpringerPlus

RESEARCH Open Access

Lyapunov-type inequality for a higher order dynamic equation on time scales

Taixiang Sun[1,2*] and Hongjian Xi[1,2]

*Correspondence: stx1963@163.com
[2] Colleges and Universities Key Laboratory of Mathematics and Its Applications, Nanning 530004, China
Full list of author information is available at the end of the article

Abstract

The purpose of this work is to establish a Lyapunov-type inequality for the following dynamic equation

$$S_n^{\Delta}(t,x(t)) + u(t)x^p(t) = 0$$

on some time scale $\mathbf{T}$ under the anti-periodic boundary conditions $S_k(a,x(a))+S_k(b,x(b)) = 0$ $(0 \le k \le n-1)$, where $S_0(t,x(t)) = x(t)$, $S_k(t,x(t)) = a_k(t)S_{k-1}^{\Delta}(t,x(t))$ for $1 \le k \le n-1$ and $S_n(t,x(t)) = a_n(t)[S_{n-1}^{\Delta}(t,x(t))]^p$, $a_k \in C_{rd}(\mathbf{T},(-\infty,0)\cup(0,\infty))$ $(1 \le k \le n)$ with $a_n(a) = a_n(b)$ and $u \in C_{rd}(\mathbf{T},\mathbf{R})$, p is the quotient of two odd positive integers and $a, b \in \mathbf{T}$ with $a < b$.

Keywords: Lyapunov-type inequality, Dynamic equation, Time scale

Mathematics Subject Classification: 34K11, 34N05, 39A10

Background

Lyapunov (1907) studied the following linear differential equation

$$x''(t) + q(t)x(t) = 0 \tag{1}$$

and showed that if $q \in C([a,b],\mathbf{R})$ and $x(t) \not\equiv 0$ $(t \in [a,b])$ is a solution of (1) with $x(a) = x(b) = 0$, then the following classical Lyapunov inequality holds:

$$\int_a^b |q(t)|dt > \frac{4}{b-a},$$

Moreover, the above inequality is optimal.

Cheng (1983) investigated the following second-order difference equation

$$\Delta^2 x(n) + q(n)x(n+1) = 0 \tag{2}$$

and showed that if $x(n) \not\equiv 0$ for $n \in \{a, a+1, \ldots, b\}$ is a solution of (2) and $x(a) = x(b) = 0$ $(a, b \in \mathbf{Z}$ with $0 < a < b)$, then $\sum_{n=a}^{b-2} |q(n)| \ge \frac{4(b-a)}{(b-a)^2-1}$ if $b-a-1$ is even and $\sum_{n=a}^{b-2} |q(n)| \ge \frac{4}{b-a}$ if $b-a-1$ is odd.

Hilger (1990) introduced the theory of time scales with one goal being the unified treatment of differential equations (the continuous case) and difference equations (the discrete case). A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of the real numbers

R, which has the topology that it inherits from the standard topology on **R**. The two most popular examples are **R** and the integers **Z**. For the time scale calculus and some related basic concepts, we refer the readers to the books by Bohner and Peterson (2001, 2003) for further details.

Bohner et al. (2002) investigated the following Sturm–Liouville dynamic equation

$$x^{\Delta^2}(t) + q(t)x^{\sigma}(t) = 0 \tag{3}$$

on time scale $\mathbf{T}$ under the assumptions $x(a) = x(b) = 0$ $(a, b \in \mathbf{T}$ with $a < b)$ and $q \in C_{rd}(\mathbf{T}, (0, \infty))$ and showed if $x(t) \not\equiv 0$ for $t \in [a, b]_{\mathbf{T}}$ is a solution of (3), then

$$\int_a^b q(t)\Delta t \geq \frac{b-a}{C},$$

where $C = \max\{(t-a)(b-t) : t \in [a, b]_{\mathbf{T}}\}$.

Wong et al. (2006) investigated the following dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + q(t)x^{\sigma}(t) = 0, \tag{4}$$

on time scale $\mathbf{T}$ under the assumptions $x(a) = x(b) = 0$ $(a, b \in \mathbf{T}$ with $a < b)$ and $r \in C_{rd}([a, b]_{\mathbf{T}}, \mathbf{R})$ is monotone and $q \in C_{rd}([a, b]_{\mathbf{T}}, (0, \infty))$, and showed that if $x(t) \not\equiv 0$ for $t \in [a, b]_{\mathbf{T}}$ is a solution of (4), then

$$\int_a^b \max\{q(t), 0\}\Delta t \geq \begin{cases} \frac{r(a)(b-a)}{r(b)C}, & \text{if} \quad r \text{ is increasing}, \\ \frac{r(b)(b-a)}{r(a)C}, & \text{if} \quad r \text{ is decreasing}, \end{cases}$$

where $C = \max\{(t-a)(b-t) : t \in [a, b]_{\mathbf{T}}\}$.

In this paper, we establish a Lyapunov-type inequality for the following higher order dynamic equation

$$S_n^{\triangle}(t, x(t)) + u(t)x^p(t) = 0 \tag{5}$$

on some time scale $\mathbf{T}$ under the following anti-periodic boundary conditions

$$S_k(a, x(a)) + S_k(b, x(b)) = 0 \quad (0 \leq k \leq n-1), \tag{6}$$

where $S_0(t, x(t)) = x(t)$, $S_k(t, x(t)) = a_k(t)S_{k-1}^{\triangle}(t, x(t))$ for $1 \leq k \leq n-1$ and $S_n(t, x(t)) = a_n(t)[S_{n-1}^{\Delta}(t, x(t))]^p$, $a_k \in C_{rd}(\mathbf{T}, (-\infty, 0) \cup (0, \infty))$ $(1 \leq k \leq n)$ with $a_n(a) = a_n(b)$ and $u \in C_{rd}(\mathbf{T}, \mathbf{R})$, p is the quotient of two odd positive integers and $a, b \in \mathbf{T}$ with $a < b$.

For some other related results on Lyapunov inequality, see, for example, Çakmak (2013), He et al. (2011), Jiang and Zhou (2005), Liu and Tang (2014), Tang and Zhang (2012) and Yang et al. (2014).

Main result and its proof

Lemma 1 *(Bohner and Peterson 2001)* *Let* $a, b \in \mathbf{T}$ *with* $a < b$ *and* $\sum_{i=1}^{n} 1/p_i = 1$ *with* $p_i > 1$ $(1 \leq i \leq n)$. *Then for any functions* $f_i \in C_{rd}([a, b]_{\mathbf{T}}, \mathbf{R})$ $(1 \leq i \leq n)$, *we have*

$$\int_a^b \prod_{i=1}^n |f_i(t)|\Delta t \leq \prod_{i=1}^n \left\{ \int_a^b |f_i(t)|^{p_i} \Delta t \right\}^{\frac{1}{p_i}}.$$

Lemma 2 *Let $a, b \in \mathbf{T}$ with $a < b$. Suppose that $\alpha_i^j \in \mathbf{R}$ and $p_i \in (1, +\infty)$ with $\sum_{i=1}^n \alpha_i^j / p_i = \sum_{i=1}^n 1/p_i = 1$ $(1 \leq i \leq n, 1 \leq j \leq m)$. Then for any functions $f_j \in C_{rd}([a,b]_{\mathbf{T}}, (-\infty, 0) \cup (0, \infty))$ $(1 \leq j \leq m)$, we have*

$$\int_a^b \prod_{j=1}^m |f_j(t)|\Delta t \leq \prod_{i=1}^n \left\{ \int_a^b \prod_{j=1}^m |f_j(t)|^{\alpha_i^j} \Delta t \right\}^{\frac{1}{p_i}}.$$

Proof Let $F_i(t) = (\prod_{j=1}^m |f_j(t)|^{\alpha_i^j})^{\frac{1}{p_i}}$. By Lemma 1 we have

$$\begin{aligned}
\int_a^b \prod_{j=1}^m |f_j(t)|\Delta t &= \int_a^b \prod_{i=1}^n F_i(t)\Delta t \\
&\leq \prod_{i=1}^n \left\{ \int_a^b F_i^{p_i} \Delta t \right\}^{\frac{1}{p_i}} \\
&= \prod_{i=1}^n \left\{ \int_a^b \prod_{j=1}^m |f_j(t)|^{\alpha_i^j} \Delta t \right\}^{\frac{1}{p_i}}.
\end{aligned}$$

This completes the proof of Lemma 2. □

Remark 3 Let $i = j$, and $\alpha_i^i = p_i$ and $\alpha_i^j = 0$ if $i \neq j$ in Lemma 2, we obtain Lemma 1.

Theorem 4 *Let $\alpha_i \in \mathbf{R}$ $(1 \leq i \leq n)$, $p_1 = p + 1$ and $p_j \in (1, +\infty)$ $(2 \leq j \leq n)$ with $\sum_{i=1}^n \alpha_i / p_i = \sum_{i=1}^n 1/p_i = 1$. If (5) has a solution $x(t) \not\equiv 0$ for $t \in [a,b]_{\mathbf{T}}$ satisfying the anti-periodic boundary conditions (6), then*

$$\int_a^b |u(t)|^{\frac{p+1}{p}} \Delta t \geq \frac{2^{\frac{[(n-1)p+1](p+1)}{p}}}{(b-a)^{\frac{1}{p}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}} \right]^{p+1} \prod_{i=1}^{n-1} \left\{ \prod_{j=1}^n \left[\int_a^b \frac{\Delta t}{|a_i(t)|^{\alpha_i}} \right]^{\frac{1}{p_j}} \right\}^{p+1}}.$$

Proof For any $1 \leq i \leq n-1$, write

$$w_i = \prod_{j=1}^n \left[\int_a^b \frac{\Delta t}{|a_i(t)|^{\alpha_i}} \right]^{\frac{1}{p_j}}$$

and

$$u_i = \prod_{j=1}^n \left[\int_a^b \frac{|S_i(t, x(t))|}{|a_i(t)|^{\alpha_j}} \Delta t \right]^{\frac{1}{p_j}}.$$

Since $x(t)$ satisfies $S_i(a, x(a)) + S_i(b, x(b)) = 0$ $(0 \leq i \leq n-1)$, we know that for any $t \in [a,b]_{\mathbf{T}}$,

$$S_i(t) = S_i(a, x(a)) + \int_a^t \frac{S_{i+1}(\tau, x(\tau))}{a_{i+1}(\tau)} \Delta\tau = S_i(b, x(b)) - \int_t^b \frac{S_{i+1}(\tau, x(\tau))}{a_{i+1}(\tau)} \Delta\tau.$$

Using Lemma 2, we obtain that for $0 \le i \le n-2$,

$$\begin{aligned} |S_i(t, x(t))| &= \frac{1}{2} |S_i(a, x(a)) + \int_a^t \frac{S_{i+1}(\tau, x(\tau))}{a_{i+1}(\tau)} \Delta\tau + S_i(b, x(b)) - \int_t^b \frac{S_{i+1}(\tau, x(\tau))}{a_{i+1}(\tau)} \Delta\tau| \\ &\le \frac{1}{2} \int_a^b |\frac{S_{i+1}(t, x(t))}{a_{i+1}(t)}| \Delta t \le \frac{1}{2} u_{i+1}. \end{aligned} \tag{7}$$

and

$$\begin{aligned} |S_{n-1}(t, x(t))| &\le \frac{1}{2} \int_a^b \frac{|a_n(t)|^{\frac{1}{p_1}}}{|a_n(t)|^{\frac{1}{p_1}}} |S_{n-1}^{\Delta}(t, x(t))| \Delta t \\ &\le \frac{1}{2} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}} \right]^{\frac{p}{p_1}} \left[\int_a^b |a_n(t)||S_{n-1}^{\Delta}(t, x(t))|^{p_1} \Delta t \right]^{\frac{1}{p_1}} \quad (t \in [a, b]_{\mathbf{T}}) \end{aligned}$$

and

$$|S_{n-1}(\sigma(t), x(\sigma(t)))| \le \frac{1}{2} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}} \right]^{\frac{p}{p_1}}$$
$$\left[\int_a^b |a_n(t)||S_{n-1}^{\Delta}(t, x(t))|^{p_1} \Delta t \right]^{\frac{1}{p_1}} \quad (t \in [a, b)_{\mathbf{T}}),$$

which implies

$$\begin{aligned} u_i &= \prod_{j=1}^{n} \left[\int_a^b \frac{|S_i(t, x(t))|}{|a_i(t)|^{\alpha_j}} \Delta t \right]^{\frac{1}{p_j}}. \\ &\le \frac{1}{2} u_{i+1} w_i \quad (1 \le i \le n-2) \end{aligned} \tag{8}$$

and

$$|S_{n-1}(t, x(t))|^{p_1} \le \frac{1}{2^{p_1}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}} \right]^p \int_a^b |a_n(t)||S_{n-1}^{\Delta}(t, x(t))|^{p_1} \Delta t \quad (t \in [a, b]_{\mathbf{T}}) \tag{9}$$

and

$$|S_{n-1}(\sigma(t), x(\sigma(t)))|^{p_1} \le \frac{1}{2^{p_1}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}} \right]^p \int_a^b |a_n(t)||S_{n-1}^{\Delta}(t, x(t))|^{p_1} \Delta t \quad (t \in [a, b)_{\mathbf{T}}). \tag{10}$$

Combining (7), (8) and (9), it follows

$$|x(t)| \le M \equiv \frac{\prod_{i=1}^{n-1} w_i}{2^{n-1}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}} \right]^{\frac{p}{p_1}} \left[\int_a^b |a_n(t)||S_{n-1}^{\Delta}(t, x(t))|^{p_1} \Delta t \right]^{\frac{1}{p_1}}. \tag{11}$$

From (1), we have

$$S_n^{\Delta}(t,x(t)) = -u(t)x(t)^p.$$

Thus, we obtain

$$S_n^{\Delta}(t,x(t))S_{n-1}^{\sigma}(t,x(t)) = -u(t)x^p(t)S_{n-1}^{\sigma}(t,x(t)). \tag{12}$$

Integrating (12) from a to b, it follows

$$\int_a^b S_n^{\Delta}(t,x(t))S_{n-1}^{\sigma}(t,x(t))\Delta t = \int_a^b -u(t)x^p(t)S_{n-1}^{\sigma}(t,x(t))\Delta t. \tag{13}$$

Thus, we obtain from (10), (11) and (13) that

$$\begin{aligned}
&\int_a^b a_n(t)|S_{n-1}^{\Delta}(t,x(t))|^{p+1}\Delta t \\
&= \int_a^b a_n(t)(S_{n-1}^{\Delta}(t,x(t)))^{p+1}\Delta t \\
&= \int_a^b [(S_n(t,x(t))S_{n-1}(t,x(t)))^{\Delta} - S_n^{\Delta}(t,x(t))S_{n-1}^{\sigma}(t,x(t))]\Delta t \\
&= a_n(b)S_{n-1}^{p}(b,x(b))S_{n-1}(b,x(b)) - a_n(a)S_{n-1}^{p}(a,x(a))S_{n-1}(a,x(a)) \\
&\quad - \int_a^b S_n^{\Delta}(t,x(t))S_{n-1}^{\sigma}(t,x(t))\Delta t \\
&\le \int_a^b |u(t)x^p(t)S_{n-1}^{\sigma}(t,x(t))|\Delta t \\
&\le M^p \int_a^b |u(t)||S_{n-1}(\sigma(t),x(\sigma(t)))|\Delta t \\
&\le M^p \left[\int_a^b |u(t)|^{\frac{p_1}{p}}\Delta t\right]^{\frac{p}{p_1}} \left[\int_a^b |S_{n-1}(\sigma(t),x(\sigma(t)))|^{p_1}\Delta t\right]^{\frac{1}{p_1}} \\
&\le M^p \left[\int_a^b |u(t)|^{\frac{p_1}{p}}\Delta t\right]^{\frac{p}{p_1}} \frac{(b-a)^{\frac{1}{p_1}}}{2} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}}\right]^{\frac{p}{p_1}} \left[\int_a^b |a_n(t)||S_{n-1}^{\Delta}(t,x(t))|^{p_1}\Delta t\right]^{\frac{1}{p_1}} \\
&= \left\{\frac{\prod_{i=1}^{n-1} w_i}{2^{n-1}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}}\right]^{\frac{p}{p_1}} \left[\int_a^b |a_n(t)||S_{n-1}^{\Delta}(t,x(t))|^{p_1}\Delta t\right]^{\frac{1}{p_1}}\right\}^p \\
&\quad \times \left[\int_a^b |u(t)|^{\frac{p_1}{p}}\Delta t\right]^{\frac{p}{p_1}} \frac{(b-a)^{\frac{1}{p_1}}}{2} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}}\right]^{\frac{p}{p_1}} \left[\int_a^b |a_n(t)||S_{n-1}^{\Delta}(t,x(t))|^{p_1}\Delta t\right]^{\frac{1}{p_1}} \\
&= \frac{[\prod_{i=1}^{n-1} w_i]^p}{2^{(n-1)p+1}}(b-a)^{\frac{1}{p_1}} \left[\int_a^b |u(t)|^{\frac{p_1}{p}}\Delta t\right]^{\frac{p}{p_1}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}}\right]^p \\
&\quad \times \left[\int_a^b |a_n(t)||S_{n-1}^{\Delta}(t,x(t))|^{p+1}\Delta t\right]^{\frac{p+1}{p+1}}.
\end{aligned}$$

Since $x(t) \not\equiv 0\ (t \in [a,b]_{\mathbf{T}})$, it follows from (11) that

$$\int_a^b |a_n(t)||S_{n-1}^{\Delta}(t,x(t))|^{p+1}\Delta t > 0.$$

Thus, we obtain

$$\int_a^b |u(t)|^{\frac{p+1}{p}} \Delta t \geq \frac{2^{\frac{[(n-1)p+1](p+1)}{p}}}{(b-a)^{\frac{1}{p}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}}\right]^{p+1} \prod_{i=1}^{n-1} \left\{\prod_{j=1}^{n} \left[\int_a^b \frac{\Delta t}{|a_i(t)|^{\alpha_i}}\right]^{\frac{1}{p_j}}\right\}^{p+1}}.$$

This completes the proof of Theorem 4. □

Let $\alpha_i = 1 + r_i p_i$ $(1 \leq i \leq n)$ in Theorem 4, we obtain the following corollary.

Corollary 5 *Let* $r_i \in \mathbf{R}$ $(1 \leq i \leq n)$, $p_1 = p + 1$ *and* $p_j \in (1, +\infty)$ $(2 \leq j \leq n)$ *with* $\sum_{i=1}^{n} 1/p_i = 1$ *and* $\sum_{i=1}^{n} r_i = 0$. *If* (5) *has a solution* $x(t) \not\equiv 0$ for $t \in [a, b]_{\mathbf{T}}$ *satisfying the anti-periodic boundary conditions* (6), *then*

$$\int_a^b |u(t)|^{\frac{p+1}{p}} \Delta t \geq \frac{2^{\frac{[(n-1)p+1](p+1)}{p}}}{(b-a)^{\frac{1}{p}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}}\right]^{p+1} \prod_{i=1}^{n-1} \left\{\prod_{j=1}^{n} \left[\int_a^b \frac{\Delta t}{a_i^{1+r_i p_i}(t)}\right]^{\frac{1}{p_j}}\right\}^{p+1}}.$$

Set $\alpha_i = 1$ $(1 \leq i \leq n)$ *in Theorem* 4, *we obtain the following Corollary* 6.

Corollary 6 *If* (5) *has a solution* $x(t) \not\equiv 0$ for $t \in [a, b]_{\mathbf{T}}$ *satisfying the anti-periodic boundary conditions* (6), *then*

$$\int_a^b |u(t)|^{\frac{p+1}{p}} \Delta t \geq \frac{2^{\frac{[(n-1)p+1](p+1)}{p}}}{(b-a)^{\frac{1}{p}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}}\right]^{p+1} \prod_{i=1}^{n-1} \left[\int_a^b \frac{\Delta t}{a_i(t)}\right]^{p+1}}.$$

Examples and applications

Example 1 Suppose that $\alpha_i \in \mathbf{R}$ $(1 \leq i \leq n)$, $p_1 = p + 1$ and $p_j \in (1, +\infty)$ $(2 \leq j \leq n)$ with $\sum_{i=1}^{n} \alpha_i/p_i = \sum_{i=1}^{n} 1/p_i = 1$. Let $\mathbf{T} = [-2, -1] \cup [1, \infty)$, $a_k(t) = t$ for $1 \leq k \leq n-1$ and $a_n = t^{2m}$ for some positive integer m, and

$$u(t) = \begin{cases} -(2m+1)2m(p+1)/t^{p+1-2m}, & \text{if } t \neq -1, \\ \{(2m+1)^{np} - [1/2^{n-1} + \sum_{i=1}^{n-1}(2m+1)^{n-i}/2^i]^p]\}/2, & \text{if } t = -1. \end{cases}$$

Set $x(t) = t^{2m+1}$. It is easy to check that

(1) $S_k(t, x(t)) = (2m+1)^k t^{2m+1}$ $(0 \leq k \leq n-1)$, $S_n(t, x(t)) = (2m+1)^{np} t^{2m(p+1)}$ and $S_n^{\triangle}(t, x(t)) = (2m+1)^{np} 2m(p+1) t^{2m(p+1)-1}$ for $t \neq -1$.

(2) $S_0(-1, x(-1)) = S_1(-1, x(-1)) = -1$, $S_k(-1, x(-1)) = -[1/2^{k-1} + \sum_{i=1}^{k-1}(2m+1)^{k-i}/2^i]$ $(0 \leq k \leq n-1)$, $S_n(-1, x(-1)) = [1/2^{n-1} + \sum_{i=1}^{n-1}(2m+1)^{n-i}/2^i]^p$ and $S_n^{\triangle}(-1, x(-1)) = \{(2m+1)^{np} - [1/2^{n-1} + \sum_{i=1}^{n-1}(2m+1)^{n-i}/2^i]^p]\}/2$. Let $a = -2$ and $b = 2$. Then $x(t) \not\equiv 0$ is a solution of (5) satisfying the anti-periodic boundary conditions (6). Thus we have

$$\int_{-2}^{2} |u(t)|^{\frac{p+1}{p}} \Delta t \geq \frac{2^{(n-1)(p+1)+1-\frac{1}{p}}}{\left[\int_{-2}^{2} \frac{\Delta t}{|t|^{\frac{2m}{p}}}\right]^{p+1} \prod_{i=1}^{n-1} \left\{\prod_{j=1}^{n} \left[\int_{-2}^{2} \frac{\Delta t}{|t|^{\alpha_i}}\right]^{\frac{1}{p_j}}\right\}^{p+1}}.$$

Example 2 Suppose that $\alpha_i \in \mathbf{R}\,(1 \leq i \leq n)$, $p_1 = p+1$ and $p_j \in (1, +\infty)\,(2 \leq j \leq n)$ with $\sum_{i=1}^{n} \alpha_i/p_i = \sum_{i=1}^{n} 1/p_i = 1$. Let $\mathbf{T} = \{\pm 2^n : n = 0, 1, 2, \ldots\}$, $a_k(t) = t$ for $1 \leq k \leq n-1$ and $a_n = t^2$ and $u(t) = -(\sigma(t) + t)/t^p$. Write $x(t) = t$. It is easy to check that $S_k(t, x(t)) = t\,(0 \leq k \leq n-1)$, $S_n(t, x(t)) = t^2$ and $S_n^{\triangle}(t, x(t)) = \sigma(t) + t$.

Let $a = -2^r$ and $b = 2^r$ for some positive integer r. Then $x(t) \not\equiv 0$ is a solution of (5) satisfying the anti-periodic boundary conditions (6). Thus we have

$$\int_{-2^r}^{2^r} \left|\frac{\sigma(t) + t}{t^p}\right|^{\frac{p+1}{p}} \Delta t \geq \frac{2^{(n-1)(p+1)+1-\frac{r}{p}}}{\left[\int_{-2^r}^{2^r} \frac{\Delta t}{|t|^{\frac{2}{p}}}\right]^{p+1} \prod_{i=1}^{n-1} \left\{\prod_{j=1}^{n} \left[\int_{-2^r}^{2^r} \frac{\Delta t}{|t|^{\alpha_i}}\right]^{\frac{1}{p_j}}\right\}^{p+1}}.$$

Now, we give an application of Lyapunov-type inequality of Theorem 4 for the following eigenvalue problem

$$S_n^{\triangle}(t, x(t)) + ru(t)x^p(t) = 0 \tag{14}$$

on time scale $[a, b]_{\mathbf{T}}$ for some $a, b \in \mathbf{T}$ with $a < b$, where $S_0(t, x(t)) = x(t)$, $S_k(t, x(t)) = a_k(t) S_{k-1}^{\triangle}(t, x(t))$ for $1 \leq k \leq n-1$ and $S_n(t, x(t)) = a_n(t)[S_{n-1}^{\Delta}(t, x(t))]^p$, $a_k \in C_{rd}([a, b]_{\mathbf{T}}, (-\infty, 0) \cup (0, \infty))\,(1 \leq k \leq n)$ with $a_n(a) = a_n(b)$ and $u \in C_{rd}([a, b]_{\mathbf{T}}, \mathbf{R})$, p is the quotient of two odd positive integers. It is easy to see the lower bound of the eigenvalue r in (14)

$$|r| \geq \frac{2^{(n-1)p+1}}{\left[\int_a^b |u(t)|^{\frac{p+1}{p}} \Delta t\right]^{\frac{p}{p+1}} (b-a)^{\frac{1}{p+1}} \left[\int_a^b \frac{\Delta t}{|a_n(t)|^{\frac{1}{p}}}\right]^p \prod_{i=1}^{n-1} \left\{\prod_{j=1}^{n} \left[\int_a^b \frac{\Delta t}{|a_i(t)|^{\alpha_i}}\right]^{\frac{1}{p_j}}\right\}^p},$$

where $\alpha_i \in \mathbf{R}\,(1 \leq i \leq n)$, $p_1 = p+1$ and $p_j \in (1, +\infty)\,(2 \leq j \leq n)$ with $\sum_{i=1}^{n} \alpha_i/p_i = \sum_{i=1}^{n} 1/p_i = 1$.

Conclusions

In this paper, we establish a Lyapunov-type inequality for the following higher order dynamic equation

$$S_n^{\triangle}(t, x(t)) + u(t)x^p(t) = 0$$

on some time scale $\mathbf{T}$ under the anti-periodic boundary conditions (6). Our results complement with some previous ones.

Authors' contributions

All authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Author details

[1] College of Information and Statistics, Guangxi University of Finance and Economics, Nanning 530003, China. [2] Colleges and Universities Key Laboratory of Mathematics and Its Applications, Nanning 530004, China.

Acknowledgements
This project is supported by NNSF of China (11461003) and NSF of Guangxi (2014GXNSFBA118003).

Competing interests
The authors declare that they have no competing interests.

Received: 30 November 2015 Accepted: 23 August 2016
Published online: 01 September 2016

References

Bohner M, Clark S, Ridenhour J (2002) Lyapunov inequalities for time scales. J Inequal Appl 7:61–77
Bohner M, Peterson A (2001) Dynamic equations on time scales: an introduction with applications. Birkhauser, Boston
Bohner M, Peterson A (2003) Advances in dynamic equations on time scales. Birkhauser, Boston
Çakmak D (2013) Lyapunov-type inequalities for two classes of nonlinear systems with anti-periodic boundary conditions. Appl Math Comput 223:237–242
Cheng SS (1983) A discrete analogue of the inequality of Lyapunov. Hokkaido Math J 12:105–112
He X, Zhang Q, Tang X (2011) On inequalities of Lyapunov for linear Hamiltonian systems on time scales. J Math Anal Appl 381:695–705
Hilger S (1990) Analysis on measure chains—a unified approach to continuous and discrete calculus. Results Math 18:18–56
Jiang L, Zhou Z (2005) Lyapunov inequality for linear Hamiltonian systems on time scales. J Math Anal Appl 310:579–593
Liu X, Tang M (2014) Lyapunov-type inequality for higher order difference equations. Appl Math Comput 232:666–669
Lyapunov AM (1907) Probléme génèral de stabilité du mouvement. Ann Fac Sci Toulouse Math 9:203–474
Tang X, Zhang M (2012) Lyapunov inequalities and stability for linear Hamiltonian systems. J Differ Equ 252:358–381
Wong F, Yu S, Yeh C, Lian W (2006) Lyapunov's inequality on time scales. Appl Math Lett 19:1293–1299
Yang X, Kim Y, Lo K (2014) Lyapunov-type inequality for a class of even-order linear differential equations. Appl Math Comput 245:145–151