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A characterization of some alternating groups A_{p+8} of degree $p + 8$ by OD

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Abstract

Let A_n be an alternating group of degree n. We know that A_{10} is 2-fold *OD*-characterizable and A_{125} is 6-fold *OD*-characterizable. In this note, we first show that A_{189} and A_{147} are 14-fold and 7-fold OD-characterizable, respectively, and second show that certain groups A_{p+8} with that $\pi((p+8)!) = \pi(p!)$ and $p < 1000$, are OD-characterizable. The first gives a negative answer to Open Problem of Kogani-Moghaddam and Moghaddamfar.

Keywords: Element order, Alternating group, Simple group, Symmetric group, Degree pattern, Prime graph

Mathematics Subject Classification: 20D05, 20D06, 20D60

Background

For a group, it means finite, and for a simple group, it is non-abelian. If G is a group, then the set of element orders of G is denoted by $\omega(G)$ and the set of prime divisors of G is denoted by $\pi(G)$. Related to the set $\omega(G)$ a graph is named a prime graph of G, which is written by $GK(G)$. The vertex set of $GK(G)$ is written by $\pi(G)$, and for different primes p, q, there is an edge between the two vertices p, q if $p \cdot q \in \omega(G)$, which is written by $p \sim q$. We let $s(G)$ denote the number of connected components of the prime graph $GK(G)$.

Moghaddamfar et al in 2005 gave the following notions which inspire some authors' attention.

Definition 1 (Moghaddamfar et al. 2005) Let G be a finite group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_is are primes and α_is are positive integers. For $p \in \pi(G)$, let $\deg(p) := |\{q \in \pi(G) | p \sim q\}|$, which we call the degree of p. We also define $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, where $p_1 < p_2 < \cdots < p_k$. We call $D(G)$ the degree pattern of G.

For a given finite group M, write $h_{OD}(M)$ to denote the number of isomorphism classes of finite groups G such that (1) $|G| = |M|$ and (2) $D(G) = D(M)$.

Definition 2 (Moghaddamfar et al. 2005) A finite group M is called k-fold OD-characterizable if $h_{OD}(M) = k$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable group.

Up to now, some groups are proved to be k-fold OD-characterizable and we can refer to the corresponding references of Akbari and Moghaddamfar (2015).

Concerning the alternating group G with $s(G) = 1$, what's the influence of OD on the structure of group? Recently, the following results are given.

Theorem 3 *The following statements hold:*

1. *The alternating group A_{10} is 2-fold OD-characterizable* (*see* Moghaddamfar and Zokayi 2010).
2. *The alternating group A_{125} is 6-fold OD-characterizable (see* Liu and Zhang Submitted).
3. *The alternating group A_{p+3} except A_{10} is OD-characterizable (see* Hoseini and Moghaddamfar 2010; Kogani-Moghaddam and Moghaddamfar 2012; Liu 2015; Moghaddamfar and Rahbariyan 2011; Moghaddamfar and Zokayi 2009; Yan and Chen 2012; Yan et al. 2013; Zhang and Shi 2008; Mahmoudifar and Khosravi 2015).
4. *All alternating groups A_{p+5}, where $p + 4$ is a composite and $p + 6$ is a prime and $5 \neq p \in \pi(1000!)$, are OD-characterizable (see* Yan et al. 2015).

In Moghaddamfar (2015), *A_{189} is at least 14-fold OD-characterizable. In this paper, we show the results as follows.*

Theorem 4 *The following hold:*

1. *The alternating group A_{189} of degree 189 is 14-fold OD-characterizable.*
2. *The alternating group A_{147} of degree 147 is 7-fold OD-characterizable.*

These results give negative answers to the Open Problem (Kogani-Moghaddam and Moghaddamfar 2012).

Open Problem (Kogani-Moghaddam and Moghaddamfar 2012) All alternating groups A_m, with $m \neq 10$, are OD-characterizable.

We also prove that some alternating groups A_{p+8} with $p < 1000$ are OD-characterizable.

Theorem 5 *Assume that p is a prime satisfying the following three conditions:*

1. $p \neq 139$ *and* $p \neq 181$,
2. $\pi((p + 8)!) = \pi(p!)$,
3. $p \leq 997$.

Then the alternating group A_{p+8} of degree $p + 8$ is OD-characterizable.

Let G be a finite group, then let $\mathrm{Soc}(G)$ denote the socle of G regarded as a subgroup which is generated by the minimal normal subgroup of G. Let $\mathrm{Syl}_p(G)$ be the set of all Sylow p-subgroups G_p of G, where $p \in \pi(G)$. Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ be the automorphism and outer-automorphism group of G, respectively. Let S_n denote the symmetric groups of degree n. Let p be a prime divisor of a positive integer n, then the p-part of n is denoted by n_p, namely, $n_p \| n$. The other symbols are standard (see Conway et al. 1985, for instance).

Some preliminary results

In this section, some preliminary results are given to prove the main theorem.

Lemma 6 *Let $S = P_1 \times \cdots \times P_r$, where P_i's are isomorphic non-abelian simple groups. Then* $\mathrm{Aut}(S) = \mathrm{Aut}(P_1) \times \cdots \times \mathrm{Aut}(P_r).S_r$.

Proof See Zavarnitsin (2000). □

Lemma 7 *Let A_n (or S_n) be an alternating (or a symmetric group) of degree n. Then the following hold.*

(1) *Let $p, q \in \pi(A_n)$ be odd primes. Then $p \sim q$ if and only if $p + q \leq n$.*
(2) *Let $p \in \pi(A_n)$ be odd prime. Then $2 \sim p$ if and only if $p + 4 \leq n$.*
(3) *Let $p, q \in \pi(S_n)$. Then $p \sim q$ if and only if $p + q \leq n$.*

Proof It is easy to get from Zavarnitsin and Mazurov (1999). □

Lemma 8 *The number of groups of order 189 is 13.*

Proof See Western (1898). □

Lemma 9 *Let P be a finite simple group and assume that r is the largest prime divisor of $|P|$ with $50 < r < 1000$. Then for every prime number s satisfying the inequality $(r-1)/2 < s \leq r$, the order of the factor group $\mathrm{Aut}(P)/P$ is not divisible by s.*

Proof It is easy to check this results by Conway et al. (1985) and Zavarnitsine (2009). □

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ where $p_1, p_2, \ldots, p_r$ are different primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers, then $\exp(n, p_i) = \alpha_i$ with $p_i^{\alpha_i} \mid n$ but $p_i^{\alpha_i+1} \nmid n$.

Lemma 10 *Let $L := A_{p+8}$ be an alternating group of degree $p+8$ with that p is a prime and $\pi(p+8)! = \pi(p!)$. Let $|\pi(A_{p+8})| = d$ with d a positive integer. Then the following hold:*

(1) $\deg(p) = 4$ *and* $\deg(r) = d - 1$ *for* $r \in \{2, 3, 5, 7\}$.
(2) $\exp(|L|, 2) \leq p + 7$.
(3) $\exp(|L|, r) = \sum_{i=1}^{\infty} [\frac{p+8}{r^i}]$ *for each* $r \in \pi(L) \backslash \{2\}$. *Furthermore,* $\exp(|L|, r) < \frac{p+8}{2}$ where $5 \leq r \in \pi(L)$. *In particular, if* $r > [\frac{p+8}{2}]$, *then* $\exp(|L|, r) = 1$.

Proof By Lemma 7, it is easy to compute that for odd prime $r, p \cdot r \in \omega(L)$ if and only if $p + r \leq p + 8$. Hence $r = 3, 5, 7$. If $r = 2$, then since $p + 4 \leq p + 8$, then $2 \cdot p \in \omega(L)$. This completes (1).

By Gaussian's integer function,

$$\begin{aligned} \exp(|L|, 2) &= \sum_{i=1}^{\infty} \left[\frac{p+8}{2^i}\right] - 1 \\ &= \left(\left[\frac{p+8}{2}\right] + \frac{p+8}{2^2} + \left[\frac{p+8}{2^3}\right] + \cdots \right) - 1 \\ &\leq \left(\frac{p+8}{2} + \frac{p+8}{2^2} + \frac{p+8}{2^3} + \cdots \right) - 1 \\ &= p + 7. \end{aligned}$$

This proves (2). Similarly, we can get (3). □

Lemma 11 *Let a, m be positive integers. If $(a, m) = 1$, then the equation $a^x \equiv 1(\text{mod } m)$ has solutions. In particular, if the order of a modulo m is $h(a)$, then $h(a)$ divides $\phi(m)$ where $\phi(m)$ denotes the Euler's function of m.*

Proof See Theorem 8.12 of Burton (2002). □

Lemma 12 *Let p be a prime and $L := A_{p+8}$ be the alternating group of degree $p + 8$ with that $\pi((p + 8)!) = \pi(p!)$. Given $P \in \mathrm{Syl}_p(L)$ and $Q \in \mathrm{Syl}_q(L)$ with $11 \leq q < p \leq 1000$. Then the following results hold:*

(1) *The order of $N_L(P)$ is not divisible by $q^{s(q)}$, where $s(q) = \exp(|L|, q)$.*
(2) *If $p \in \{113, 139, 199, 211, 241, 283, 293, 337, 467, 509, 619, 787, 797, 839, 863, 887, 953, 997\}$, then $|N_L(Q)|$ is not divisible by p.*
(3) *If $p \in \{181, 317, 409, 421, 523, 547, 577, 631, 661, 691, 709, 811, 829, 919\}$, then there is at least a prime r with that the order of r modulo p is less than $p - 1$, where $11 \leq r < p$ and $r \in \pi(p!)$.*

Proof By Lemma 11, the equation $q^x \equiv 1(\text{mod } p)$ has solutions. Suppose the order of q modulo p is written by $h(q)$. If $h(q) = p - 1$, then q is a primitive root of modulo p. By Lemma 11, we have $h(q) \mid p - 1$. By Lemma 10, we can get $s(q)$. If $h(q) > s(q)$, then $q^{h(q)} \mid |L|$, a contradiction to the hypotheses. Then we can assume that $h(q) \leq s(q)$. We can get the q and $h(q)$ by GAP (2016) as Table 1 (Note that there is certain prime which has order $h(q) < p - 1$, but $h(q) > s(q)$. Hence we do not list in this table).

By NC Theorem, the factor group $\frac{N_L(P)}{C_L(P)}$ is isomorphic to a subgroup of $\mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$ where $\mathbb{Z}_n$ is a cyclic group of order n. It follows that the order of $\frac{N_L(P)}{C_L(P)}$ is less than or equal to $p - 1$. If $11 \leq q < p$ and $q^{s(q)} \mid |N_L(P)|$ where $\exp(|L|, q) = s(q)$, then $q \mid |C_L(P)|$. This forces $q \sim p$, a contradiction. This ends the proof of (1).

Next, assume that $p \in \{113, 139, 199, 211, 241, 283, 293, 337, 467, 509, 619, 787, 797, 839, 863, 887, 953, 997\}$. If p divides the order of $N_L(Q)$, then by NC theorem and Table 1, $p \mid |C_L(Q)|$ and so $p \sim q$, a contradiction. This proves (2). (3) follows from Table 1.

This completes the proof of Lemma 12. □

Table 1 The values of *p* and *h*(*q*)

p	*h*(*q*)	Condition	*p*	*h*(*q*)	Condition
113	$2^4.7$	None	139	2.3.23	None
181	$2^2.3^2.5$	$q \neq 19$	181	4	$q = 19$
199	$2.3^2.11$	None	211	2.3.5.7	None
241	$2^4.3.5$	None	283	2.3.47	None
293	$2^2.73$	None	317	$2^2.79$	$q \neq 73$
317	4	$q = 73$	337	$2^4.3.7$	None
409	$2^3.3.17$	$q \neq 31, 53$	409	8	$q = 31$
409	3	$q = 53$	421	$2^2.3.5.7$	$q \neq 29$
421	4	$q = 29$	467	2.233	None
509	$2^2.127$	none	523	$2.3^2.29$	$q \neq 11, 19, 61$
523	29	$q = 11$	523	9	$q = 19$
523	6	$q = 61$	547	2.3.7.13	$q \neq 11, 13, 41$
547	39	$q = 11$	547	21	$q = 13$
547	6	$q = 41$	577	$2^6.3^2$	$q \neq 23$
577	8	$q = 23$	619	2.3.103	None
631	$2.3^2.5.7$	$q \neq 43$	631	3	$q = 43$
661	$2^3.3.5.11$	$q \neq 11$	661	33	$q = 11$
691	2.3.5.23	$q \neq 89$	691	5	$q = 89$
709	$2^2.3.59$	$q \neq 227$	709	3	$q = 227$
787	2.3.131	None	797	$2^2.199$	None
811	$2.3^4.5$	$q \neq 131$	811	6	$q = 131$
829	$2^2.3^2.23$	$q \neq 11$	829	23	$q = 11$
839	2.419	None	863	2.431	None
887	2.443	None	919	$2.3^3.17$	$q \neq 53$
919	6	$q = 53$	953	$2^3.7.17$	None
997	$2^2.3.83$	None			

Proof of the main theorem

In this section, we first give the proof of Theorem 4 and second prove Theorem 5.

The proof of Theorem 4

Proof We divides the proof into two steps.

Step 1 Let $M = A_{189}$. Assume that G is a finite group such that

$$|G| = |M|$$

and

$$D(G) = D(M).$$

By Lemma 7, the degree pattern $GK(G)$ of G is connected, in particular, the degree pattern $GK(G)$ is the same as the degree pattern of $GK(M)$.

Lemma 13 *Let K be a maximal normal soluble subgroup of G. Then K is a* $\{2,3,5,7\}$ *-group, in particular, G is insoluble.*

Proof Assume the contrary. First we show that K is a $181'$-group. We assume that K contains an element x of order 181. Let C be the centralizer of x in G and N be the normalizer of x in G. It is easy to see from $D(G)$ that C is a $\{2,3,5,7,181\}$-group. By NC theorem, N/C is isomorphic to a subgroup of automorphism group $\mathrm{Aut}(\langle x\rangle)\cong \mathbb{Z}_{2^2}\times\mathbb{Z}_{3^2}\times\mathbb{Z}_5$, where $\mathbb{Z}_n$ is a cyclic group of order n. Hence, C is a $\{2,3,5,7,181\}$-group. By Frattini's arguments, $G=KN_G(\langle x\rangle)$ and so $\{11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97,101,103,107,109,113,127,131,137,139,149,151,157,163,167,173,179,181\}\subseteq\pi(K)$. Since K is soluble, G has a Hall subgroup H of order $109\cdot 181$. Obviously, $109\nmid 181-1, H$ is cyclic and so $109\cdot 181\in\omega(G)$ contradicting $D(G)=D(M)$.

Second, show that K is a p'-group, where $p\in\{11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97,101,103,107,109,113,127,131,137,139,149,151,157,163,167,173,179\}$. Let p be a prime divisor of $|K|$ and P a Sylow p-subgroup of K. By Frattini's arguments, $G=KN_G(P)$. It follows from Lemma 12, that 181 is a divisor of $|N_G(P)|$ if and only if $p=19$. If $181\nmid|\mathrm{Aut}(P)|$, then 181 divides the order of $C_G(P)$ and so there is an element of order $p\cdot 181$, a contradiction. On the other hand, $p=19$ and $181\mid|\mathrm{Aut}(P)|$, where P is the Sylow 19-subgroup of K. By Lemma 10, $\exp(|L|,19)=9$ and so $|\frac{N_G(P)}{C_G(P)}|\mid\prod_{i=1}^{9}19^{45}\cdot(19^i-1)$. It is easy to get that $101\nmid|\prod_{i=1}^{9}19^{45}\cdot(19^i-1)$. If $101\mid|N_G(P)|$, then 101 is a prime divisor of $C_G(P)$. Set $C=C_G(P)$ and $C_{101}\in\mathrm{Syl}_{101}(C)$. Also $\exp(|L|,101)=1$. By Frattini's argument, $N=CN_N(C_{101})$ and so $p\nmid|N_N(C_{101})|$. Thus $181\mid|C|$ and so $181\sim p$, a contradiction. So $101\nmid|N_G(P)|$ and $101\in\pi(K)$. Let $K_{101}\in\mathrm{Syl}_{101}(K)$. Since $G=KN_G(K_{101})$, 101 divides the order of $N_G(K_{101})$, then $101\nmid|K|$, a contradiction. Therefore K is a $\{2,3,5,7\}$-group.

Obviously, $G\neq K$ and so G is insoluble. □

Lemma 14 *The quotient group* $G\,/\,K$ *is an almost simple group. More precisely, there is a normal series such that* $S\leq G/K\leq\mathrm{Aut}(S)$, *where S is isomorphic to* A_n *for* $n\in\{181,182,183,184,185,186,187,188,189\}$.

Proof Let $H=G/K$ and $S=\mathrm{Soc}(H)$. Then $S=B_1\times\cdots\times B_n$, where B_i's are non-abelian simple groups and $S\leq H\leq\mathrm{Aut}(S)$. In what follows, we will prove that $n=1$ and $S\cong A_n$.

Suppose the contrary. Obviously, 181 does not divide the order of S, otherwise, there is an element of order $109\cdot 181$ contradicting $D(G)=D(A_{189})$. Hence, for every i, we have that $B_i\in\mathfrak{F}_{179}$, where $\mathfrak{F}_p$ is the set of non-abelian simple group S with that $p\in\pi(S)\subseteq\{2,3,\cdots,p\}$ and p is a prime. But by Lemma 13, K is a $\{2,3,5,7\}$ -group. Therefore $181\in\pi(H)\subseteq\pi(\mathrm{Aut}(S))$ and so 181 divides the order of $\mathrm{Out}(S)$. By Lemma 6, $\mathrm{Out}(S)=\mathrm{Out}(P_1)\times\cdots\times\mathrm{Out}(P_r)$, where the group P_i's are satisfying $S\cong P_1\times\cdots\times P_r$. Therefore for some j 181 divides the order of an outer-automorphism group of a direct P_j of t isomorphic simple group B_i. Since $B_i\in\mathfrak{F}_{179}$, the order of $\mathrm{Out}(B_i)$ is not divisible by 181 by Lemma 9. By Lemma 6, $|\mathrm{Aut}(P_j)|=|\mathrm{Aut}(P_j)|^t\cdot t!$. It means $t\geq 181$, and hence $4^{181}\mid|G|$, a contradiction. Thus $n=1$ and $S=B_1$.

By Lemma 13, we can assume that $|S| = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^{18} \cdot 13^{15} \cdot 17^{11} \cdot 19^9 \cdot 23^8 \cdot 29^6 \cdot 31^6 \cdot 37^5 \cdot 41^4 \cdot 43^4 \cdot 47^4 \cdot 53^4 \cdot 59^3 \cdot 61^3 \cdot 67^2 \cdot 71^2 \cdot 73^2 \cdot 79^2 \cdot 83^2 \cdot 89^2 \cdot 97 \cdot 101 \cdot 107 \cdot 109 \cdot 113 \cdot 127 \cdot 131 \cdot 137 \cdot 139 \cdot 149 \cdot 151 \cdot 157 \cdot 163 \cdot 167 \cdot 173 \cdot 179 \cdot 181$, where $2 \le a \le 182, 1 \le b \le 93, 1 \le c \le 45$ and $1 \le d \le 30$. By Zavarnitsine (2009), the only possible group is isomorphic to A_n with $n \in \{181, 182, \ldots, 189\}$.

This completes the proof. □

We continue the proof of Theorem 4. By Lemma 14, S is isomorphic to A_n with $n \in \{181, 182, \cdots, 189\}$, and $S \le G/K \le \mathrm{Aut}(S)$.

Case 1 Let $S \cong A_{181}$.

Then $A_{181} \le G/K \le S_{181}$. If $G/K \cong A_{181}$, then $|K| = 182 \cdot 183 \cdot 184 \cdot 185 \cdot 186 \cdot 187 \cdot 188 \cdot 189 = 2^6 \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 31 \cdot 37 \cdot 47$ and so $11, 13, 17, 23, 31, 37, 47 \in \pi(K)$ contradicting to Lemma 13.

If $G/K \cong S_{181}$, we also have that 11, 13, 17 or 19 divides $|K|$, contradicting to Lemma 13.

Similarly we can rule out these cases "$S \cong A_n$ with $n \in \{182, 183, \cdots, 187\}$".

Case 2 Let $S \cong A_{188}$.

Then $A_{188} \le G/K \le S_{188}$. Therefore $G/K \cong A_{188}$ or $G/K \cong S_{188}$.

(1.1) Let $G/K \cong A_{188}$. Then $|K| = 7 \cdot 3^3$. By Conway et al. (1985), the order of $\mathrm{Out}(A_{188})$ is 2 and the Schur multiplier of A_{188} is 2. Then G is isomorphic to $K \times A_{188}$. By Lemma 8, there are 13 types of groups of order 189 satisfying that $|G| = |M|$ and $D(G) = D(M)$.

(1.2) Let $G/K \cong S_{188}$. Since $|S_{188}|_2 = |S_{189}|_2 > |A_{189}|_2$, then we rule out this case.

Case 3 Let $S \cong A_{189}$.

Then $A_{189} \le G/K \le S_{189}$. If $G/K \cong A_{189}$, then order consideration implies that G is isomorphic to A_{189}. If $G/K \cong S_{189}$, then as $|S_{189}|_2 > |A_{189}|_2 = |G|_2$, we rule out this case.

Step 2 Similarly as the proof of (1), the following results are given:

(1) K is a maximal soluble normal $\{2, 3, 5, 7\}$-group.

(2) $S \le G/K \le \mathrm{Aut}(S)$, where S is isomorphic to one of the groups: $A_{139}, A_{140}, \ldots, A_{146}$ and A_{147}.

Case 1 Let $S \cong A_{139}$.

Then $A_{139} \le G/K \le S_{139}$. If the former, then $11 \mid |K|$, a contradiction. If the latter, we also have that $11 \mid |K|$ and so we rule out.

Similarly we can rule out these cases "S is isomorphic to $A_{140}, A_{141}, \ldots, A_{145}$".

Case 2 Let $S \cong A_{146}$.

Then $A_{146} \leq G/K \leq S_{146}$. If $G/K \cong A_{146}$, then $|K| = 3 \cdot 7^2$. Since the order of $\mathrm{Out}(A_{147})$ is 2 and the Schur multiplier of A_{147} is 2. Then G is isomorphic to $K \times A_{146}$. By GAP (2016), there are six types of groups of order 147. So there are 6 groups with the hypotheses: $|G| = |A_{147}|$ and $D(G) = D(A_{147})$. If $G/K \cong S_{147}$, then as $|S_{146}|_2 > |A_{146}|_2 = |A_{147}|_2 = |G|_2$, we rule out.

Case 3 Let $S \cong A_{147}$.

Then $A_{147} \leq G/K \leq S_{147}$. If the former, then $K = 1$ and so $G \cong A_{147}$, the desired result. If the latter, then as $|S_{147}|_2 > |A_{147}|_2 = |G|_2$, we rule out.

We also can get that A_{147} is 7-fold OD-characterizable.

This completes the proof of Theorem 4. □

The proof of Theorem 5

Proof Assume that $|G| = |A_{p+8}|$ and $D(G) = D(A_{p+8})$, then by Lemma 7, the degree pattern $GK(G)$ of G is the same as $GK(A_{p+8})$ of A_{p+8}. Similarly as the proof of Theorem 4, the statements are gotten:

(1) Let K be a maximal soluble group. Then K is a $\{2, 3, 5, 7\}$-group, in particular, G is insoluble.
(2) There is a normal series such that $S \leq G/K \leq \mathrm{Aut}(S)$, where S is isomorphic to A_{p+r} with that $0 \leq r \leq 8$ and $p \in \{113, 139, 199, 211, 241, 283, 293, 317, 337, 409, 421, 467, 509, 523, 547, 577, 619, 631, 661, 691, 709, 787, 797, 811, 829, 839, 863, 887, 919, 953, 997\}$.

In what follows, we consider the case "$p = 113$".

(1) $S \cong A_{113}$.
Then $A_{113} \leq G/K \leq S_{113}$. If $G/K \cong A_{113}$, then 11 divides the order of K, a contradiction. If $G/K \cong S_{113}$, then we also have that $11 \mid |K|$, a contradiction. Similarly we can get a contradiction when S is isomorphic to one of $A_{114}, A_{115}, A_{116}, A_{117}, A_{118}, A_{119}$, and A_{120}.
(2) Let $S \cong A_{121}$.
Then $A_{121} \leq G/K \cong S_{121}$. If $G/K \cong A_{121}$, then $K = 1$, the desired result. If $G/K \cong S_{121}$, then as $|S_{121}|_2 > |G|_2 = |A_{121}|_2$, a contradiction.

Similarly we can deal with these cases "$p \in \{139, 199, 211, 241, 283, 293, 317, 337, 409, 421, 467, 509, 523, 547, 577, 619, 631, 661, 691, 709, 787, 797, 811, 829, 839, 863, 887, 919, 953, 997\}$".

This completes the proof of Theorem 5. □

Non OD-characterization of some alternating groups

Assume that p is a prime and m is an integer larger than 3. If $\pi((p+m)!) \subseteq \pi(p!)$, then $GK(A_{p+m})$ is connected.

For the alternating group A_{p+m}, $|A_{p+m}| = (p+m)|A_{p+m-1}|$.

We shall use the notation $\nu(n)$ to denote the number of types of groups of order n where n is a positive integer. We follows the method of Moghaddamfar (2015), $h_{OD}(A_{p+m}) \geq 1 + \nu(p+m)$ where $\pi(A_{p+m}) = \pi(A_p)$ and $m \geq 1$ is a non-prime integer. We get the results as Table 2 which contains some results of Liu and Zhang (Submitted), Moghaddamfar (2015), Mahmoufifar and Khosravi (2014).

Note that $\nu(n)$, the number of groups of given small order n can be computed by GAP (2016). The Gap programme is as followings.

gap> SmallGroupsInformation(n);

So we have the following conjecture.

Conjecture Assume that p is a prime and $m \geq 6$ is not a prime. If $\pi((p+m)!) \subseteq \pi(p!)$ and $\pi(p+m) \subseteq \pi(m!)$, then A_{p+m} is not OD-characterizable.

Conclusion

In this paper, we have proved the following two results.

Result 1a: The alternating group A_{189} of degree 189 is 14-fold OD-characterizable.

Result 1b: The alternating group A_{147} of degree 147 is 7-fold OD-characterizable.

Result 2: Let p be a prime with the following three conditions:

(1) $p \neq 139$ and $p \neq 181$,
(2) $\pi((p+8)!) = \pi(p!)$,
(3) $p \leq 997$.

Then the alternating group A_{p+8} of degree $p+8$ is OD-characterizable.

Table 2 Non OD-characterizability of alternating groups

G	p	m!	$\pi(p+m)$	h_{OD}	References
A_{125}	113	12!	{5}	≥6	Mahmoufifar and Khosravi (2014)
A_{147}	139	8!	{3, 7}	≥7	Moghaddamfar (2015)
A_{189}	181	8!	{3, 7}	≥14	Moghaddamfar (2015)
A_{539}	523	16!	{7, 11}	≥3	Moghaddamfar (2015)
A_{625}	619	6!	{5}	≥16	Moghaddamfar (2015)
A_{875}	863	12!	{13, 67}	≥6	Moghaddamfar (2015)
A_{1029}	1019	10!	{3, 7}	≥20	
A_{1144}	1129	15!	{2, 11, 13}	≥40	
A_{1274}	1159	15!	{2, 7, 13}	≥11	
A_{1344}	1319	25!	{2, 3, 7}	≥11,721	
A_{1352}	1319	33!	{2, 13}	≥53	

Authors' contributions

SL and ZZ contributed this paper equally. Both authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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