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On the existence of positive solutions for fractional differential inclusions at resonance

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Abstract
In this paper, we discuss the existence of positive solutions for a boundary value problem of fractional differential inclusions with resonant boundary conditions. By using the Leggett–Williams theorem for coincidences of multi-valued operators due to O'Regan and Zima, results on the existence of positive solutions are established. An example is given to illustrate the efficiency of the main theorems.

Keywords: Fractional differential inclusions, Multi-valued operator, Positive solution, Resonance

Mathematics Subject Classification: 34A60, 34A08, 34B15

Background

In this article, we investigate the existence of positive solutions of fractional differential inclusions with two-point boundary conditions:

$$\begin{cases} D_{0+}^{\alpha}u(t) \in f(t,u(t)), & 0 \le t \le 1, \\ u^{(i)}(0) = 0,\ u(0) = u(1), & i = 1, 2, \ldots, n-1, \end{cases} \tag{1}$$

where $n-1 < \alpha < n, n \ge 2$, D_{0+}^{α} denotes the Caputo fractional derivative, $f : [0,1] \times \mathbb{R} \to \mathcal{F}(\mathbb{R})$, $\mathcal{F}(\mathbb{R})$ denotes the family of nonempty compact and convex subsets of $\mathbb{R}$.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order. The fractional differential equations play an important role in various fields of science and engineering, such as chemistry, biology, control theory, viscoelastic materials, signal processing, finance, life science and so on, see Kilbas et al. (2006), Samko et al. (1993), Podlubny (1999) and Orsingher and Beghin (2004).

During the last 10 years, boundary value problems for fractional differential equations are one of the most active fields in the researches of nonlinear differential equations theories. For further details, see Bai and Lü (2005), Zhang (2006), Caballero et al. (2011), Xu et al. (2009), Lin (2007) and Goodrich (2010). Meanwhile, fractional boundary value problems at resonance have been extensively studied. For some recent works on the topic, see Kosmatov (2008, 2010), Bai (2011), Bai and Zhang (2011) and Yang and Wang (2011) and references therein. It is well known that differential inclusions have proved to be valuable tools in the modeling of many realistic problems, such as economics, optimal control and so on. Recently, fractional differential inclusions have been investigated by several researchers, we refer the reader to Agarwal et al. (2010) and Chen et al. (2013).

As shown in the above mentioned works, we can see two facts. Firstly, although the boundary value problems for fractional differential equations at resonance have been studied by some authors, the existence of positive solutions to fractional differential equations at resonance are seldom considered. Secondly, there are few papers to deal with fractional differential inclusions under resonant conditions. The study of positive solutions for higher-order fractional differential inclusions under resonant conditions has yet to be initiated.

To fill this gap, we discuss the fractional differential inclusions (1) by using the Leggett–Williams theorem for coincidences of multi-valued operators due to O'Regan and Zima (2008).

The rest of this paper is organized as follows. "Preliminaries" section, we give some necessary notations, definitions and lemmas. In "Main results" section, we obtain the existence of positive solutions of (1) by Theorem 1. Finally, an example is given to illustrate our results in "Example" section.

Preliminaries

First of all, we present the necessary definitions and lemmas from fractional calculus theory. For more details, see Kilbas et al. (2006), Samko et al. (1993) and Podlubny (1999).

Definition 1 (*Kilbas et al.* 2006) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2 (*Kilbas et al.* 2006) The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n-1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 1 (Kilbas et al. 2006) *The fractional differential equation*

$$D_{0+}^{\alpha} y(t) = 0$$

has solution $y(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}, i = 0, 1, \ldots, n-1, n = [\alpha] + 1.$

Furthermore, for $y \in AC^n[0,1]$,

$$\left(I_{0+}^{\alpha} D_{0+}^{\alpha} y\right)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k$$

and

$$\left(D_{0+}^{\alpha} I_{0+}^{\alpha} y\right)(t) = y(t).$$

Lemma 2 (Kilbas et al. 2006) *The relation*

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\alpha+\beta} f(x),$$

is valid in following case: $\beta > 0,\ \alpha + \beta > 0,\ f \in L^1(a,b)$.

In the following, let us recall some definitions on Fredholm operators and cones in Banach space (see Mawhin 1979).

Let X, Y be real Banach spaces. Consider a linear mapping $L : \mathrm{dom}L \subset X \to Y$ and a nonlinear multivalued mapping $N : X \to 2^Y$. Assume that

(A1) L is a Fredholm operator of index zero, that is, $\mathrm{Im}L$ is closed and $\dim(\mathrm{Ker}L) = \mathrm{codim}(\mathrm{Im}L) < \infty$,

(A2) $N : X \to 2^Y$ is an upper semicontinuous mapping with nonempty compact convex values.

The assumption (A1) implies that there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Ker}Q = \mathrm{Im}L$. Moreover, since $\dim(\mathrm{Im}Q) = \mathrm{codim}(\mathrm{Im}L)$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$. Denote by L_p the restriction of L to $\mathrm{Ker}P \cap \mathrm{dom}L$. Clearly, L_p is an isomorphism from $\mathrm{Ker}P \cap \mathrm{dom}L$ to $\mathrm{Im}L$, we denote its inverse by $K_p : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{dom}L$. It is known that the inclusion $Lx \in Nx$ is equivalent to

$$x \in (P + JQN)x + K_P(I - Q)Nx.$$

Let C be a cone in X such that

1. $\mu x \in C$ for all $x \in C$ and $\mu \geq 0$,
2. $x, -x \in C$ implies $x = \theta$.

It is well known that C induces a partial order in X by

$$x \preceq y \quad \text{if and only if}\ y - x \in C.$$

The following property is valid for every cone in a Banach space X.

Lemma 3 *Let* C *be a cone in* X. *Then for every* $u \in C \backslash \{0\}$ *there exists a positive number* $\sigma(u)$ *such that*

$$\|x + u\| \geq \sigma(u)\|u\| \quad \text{for all}\ x \in C.$$

Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_p(I - Q)N \quad \text{and} \quad \Psi_\gamma := \Psi \circ \gamma.$$

We use the following result due to O'Regan and Zima.

Theorem 1 (O'Regan and Zima 2008) *Let C be a cone in X and let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \backslash \Omega_1) \neq \emptyset$. Assume that (A1), (A2) hold and the following assumptions hold:*

- (A3) *$QN : X \to 2^Y$ is bounded on bounded subsets of C and $K_p(I - Q)N : X \to 2^X$ be compact on every bounded subset of C,*
- (A4) *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,*
- (A5) *$Lx \notin \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap domL$ and $\lambda \in (0, 1)$,*
- (A6) $\deg\{[I - (P + JQN)\gamma]|_{\ker L}, \ker L \cap \Omega_2, 0\} \neq 0$,
- (A7) *there exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \leq \sigma(u_0)\|y\|$ for $x \in C(u_0) \cap \partial\Omega_1$ and $y \in \Psi x$, where $C(u_0) = \{x \in C : \mu u_0 \preceq x$ for some $\mu > 0\}$ and $\sigma(u_0)$ such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,*
- (A8) *$(P + JQN)\gamma(\partial\Omega_2) \subset C$,*
- (A9) *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$,*
- (A10) *$x \notin (P + JQN)\gamma x$ for $x \in \partial\Omega_2 \cap KerL$.*

Then the equation $Lx \in Nx$ has at least one solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Main results

In this section, we state our result on the existence of positive solutions for (1).

For simplicity of notation, we set

$$G(t,s) = \begin{cases} \alpha - \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha)}(1-s)^\alpha - \frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + \frac{\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha)}, & 0 \leq t < s \leq 1, \\ \alpha - \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha)}(1-s)^\alpha - \frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + \frac{\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha)}\left(\frac{t-s}{1-s}\right)^{\alpha-1}, & 0 \leq s < t \leq 1. \end{cases}$$

By the monotonicity of the function, it is easy to verify that $G(t,s) > 0, t, s \in [0, 1]$. Here, we omit the proof. Moreover, κ is a constant which satisfies

$$0 < \kappa \leq \min\left\{1, \frac{1}{\max_{t,s\in[0,1]} G(t,s)}\right\}. \tag{2}$$

Thus, we get $1 - \kappa G(t,s) > 0, t, s \in [0, 1]$.

Theorem 2 *Assume that:*

- (H1) *$f : [0, 1] \times \mathbb{R} \to \mathcal{F}(\mathbb{R})$, f(t, u) is continuous for every $u \in \mathbb{R}$, $t \in [0, 1]$,*
- (H2) *for each $r > 0$, there exists $\alpha_r \in L^1[0, 1]$ such that $|f(t, u)| \leq \alpha_r(t)$ for a.e. $t \in [0, 1]$ and every $u \in [0, r]$, where $|f(t, u)| = \sup\{|w| : w \in f(t, u)\}$,*

(H3) *there exist positive constants* $b_1, b_2, b_3, c_1, c_2, B$ *with*

$$B > \frac{c_2}{c_1} + \frac{3b_2c_2}{\alpha b_1 c_1} + \frac{3b_3}{\alpha b_1},$$

such that

$$-\kappa x \leq w \leq -c_1 x + c_2 \quad and \quad w \leq -b_1|w| + b_2 x + b_3,$$

for all $x \in [0, B]$ *and* $w \in f(t, x)$ *with* $t \in [0, 1]$,

(H4) *there exist* $b \in (0, B)$, $t_0 \in [0, 1]$, $\rho \in (0, 1]$, $\delta \in (0, 1)$ *and the function* $q \in L^1[0, 1]$, $q(t) \geq 0, t \in [0, 1]$, $h \in C\big((0, b], \mathbb{R}^+\big)$ *such that* $w(t, u) \geq q(t)h(u)$ *for* $(t, u) \in [0, 1] \times (0, b]$ *and* $w \in f(t, u)$. $\frac{h(u)}{u^\rho}$ *is non-increasing on (0, b] with*

$$\frac{h(b)}{b}\int_0^1 G(t_0, s)(1-s)^{\alpha-1}q(s)ds \geq \frac{1-\delta}{\delta^\rho}.$$

Then the problem (1) *has at least one positive solution on [0, 1].*

Proof We use the Banach space $X = Y = C[0, 1]$ with the supremum norm $\|x\| = \max_{t\in[0,1]} |x(t)|$.

Define $L : \mathrm{dom}L \to X$ and $N : X \to 2^Y$ with $\mathrm{dom}L = \big\{x \in X : D^\alpha_{0+}x(t) \in C[0, 1], x^{(i)}(0) = 0, x(0) = x(1),\ i = 1, 2, \ldots, n-1\big\}$ by

$$Lu = D^\alpha_{0+}u$$

and

$$Nu(t) = \big\{y \in Y : y(t) \in f(t, u(t)) \text{ a.e. on}[0, 1]\big\}.$$

Then the problem (1) can be written by

$$Lu \in Nu, \quad u \in \mathrm{dom}L.$$

By Lemma 1, $D^\alpha_{0+}u(t) = 0$ has solution

$$u(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, \ldots, n-1$. According to the boundary conditions of (1), we get $c_i = 0, i = 1, 2, \ldots, n-1$. Thus, we obtain

$$\mathrm{Ker}L = \big\{u \in \mathrm{dom}L : u(t) = c \in \mathbb{R}\big\}.$$

Let $y \in \mathrm{Im}L$, so there exists $u \in \mathrm{dom}L$ which satisfies $Lu = y$. By Lemma 1, we have

$$u(t) = I^\alpha_{0+}y(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}.$$

By the definition of $\mathrm{dom}L$, we have $c_i = 0,\ i = 1, 2, \ldots, n-1$. Hence,

$$u(t) = I^\alpha_{0+}y(t) + c_0.$$

Taking into account $u(0) = u(1)$, we obtain

$$\int_0^1 (1-s)^{\alpha-1}y(s)ds = 0.$$

On the other hand, suppose y satisfies the above equation. Let $u(t) = I_{0+}^{\alpha} y(t)$, and we can easily prove $u(t) \in \mathrm{dom}L$. Thus, we get

$$\mathrm{Im}L = \left\{ y \in Y : \int_0^1 (1-s)^{\alpha-1} y(s)ds = 0 \right\}.$$

Define the linear continuous projector operator $P : X \to X$ by

$$Px(t) = \alpha \int_0^1 (1-s)^{\alpha-1} x(s)ds, \quad t \in [0,1].$$

Next, we define the operator $Q : Y \to Y$ by

$$Qy(t) = \alpha \int_0^1 (1-s)^{\alpha-1} y(s)ds, \quad t \in [0,1].$$

Noting that

$$\begin{aligned} P(Px(t)) &= \alpha \int_0^1 (1-s)^{\alpha-1} Px(t)ds \\ &= Px(t) \cdot \alpha \int_0^1 (1-s)^{\alpha-1} ds \\ &= Px(t), \end{aligned}$$

then we have $P^2 = P$. Similarly, we have $Q^2 = Q$.

Then, one has $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Ker}Q = \mathrm{Im}L$. It follows from $\mathrm{Ind}L = \dim(\ker L) - \mathrm{codim}(\mathrm{Im}L) = 0$ that L is a Fredholm mapping of index zero. Then, (A1) holds.

We consider the mapping $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ by

$$K_P y(t) = \int_0^1 k(t,s) y(s)ds, \quad t \in [0,1],$$

where

$$k(t,s) := \begin{cases} \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{\Gamma(1+\alpha)}{\Gamma(2\alpha)}(1-s)^{2\alpha-1}, & 0 \le s \le t \le 1, \\ -\frac{\Gamma(1+\alpha)}{\Gamma(2\alpha)}(1-s)^{2\alpha-1}, & 0 \le t < s \le 1, \end{cases}$$

Now, we will prove that is K_P the inverse of $L|_{\mathrm{dom}} L \cap \mathrm{Ker}P$. In fact, for $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, we have $D_{0+}^{\alpha} x(t) = y(t) \in \mathrm{Im}L$ and $\alpha \int_0^1 (1-s)^{\alpha-1} x(s)ds = 0$.

By Lemma 1, one has

$$(K_P y)(t) = x(t) = I_{0+}^{\alpha} y(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}.$$

According to the definition of $\mathrm{dom}L$, we get $c_i = 0, i = 1, 2, \ldots, n-1$. Furthermore, by $\alpha \int_0^1 (1-s)^{\alpha-1} x(s)ds = 0$, we have $c_0 = -\Gamma(1+\alpha)(I_{0+}^{2\alpha} y)(1)$.

Thus,

$$
\begin{aligned}
(K_P y)(t) &= I^{\alpha}_{0+}y(t) + c_0 = I^{\alpha}_{0+}y(t) - \Gamma(1+\alpha)\left(I^{2\alpha}_{0+}y\right)(1) \\
&= I^{\alpha}_{0+}y(t) - \Gamma(1+\alpha)\cdot\frac{1}{\Gamma(2\alpha)}\int_0^1 (1-s)^{2\alpha-1}y(s)ds \\
&= \int_0^1 k(t,s)y(s)ds.
\end{aligned}
$$

Obviously, $LK_P y = y$. Moreover, for $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, we get $\alpha\int_0^1 (1-s)^{\alpha-1}x(s)ds = 0$ and

$$
\begin{aligned}
K_P Lx &= I^{\alpha}_{0+}D^{\alpha}_{0+}x(t) - \Gamma(1+\alpha)\left(I^{2\alpha}_{0+}D^{\alpha}_{0+}x\right)(1) \\
&= x(t) - x(0) - \Gamma(1+\alpha)\left(I^{\alpha}_{0+}I^{\alpha}_{0+}D^{\alpha}_{0+}x\right)(1) \\
&= x(t) - x(0) - \Gamma(1+\alpha)I^{\alpha}_{0+}(x(t)-x(0))|_{t=1} \\
&= x(t) - x(0) - \Gamma(1+\alpha)I^{\alpha}_{0+}x(1) + \Gamma(1+\alpha)I^{\alpha}_{0+}(x(0))|_{t=1} \\
&= x(t) - x(0) - \frac{\Gamma(1+\alpha)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}x(s)ds + \frac{\Gamma(1+\alpha)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}x(0)ds \\
&= x(t) - x(0) - \alpha\int_0^1 (1-s)^{\alpha-1}x(s)ds + x(0) \\
&= x(t).
\end{aligned}
$$

Thus, we know that $K_P = (L|_{\mathrm{dom}}L \cap \mathrm{Ker}P)^{-1}$. Moreover, it is easy to see that

$$
|k(t,s)| \le 3(1-s)^{\alpha-1}, \quad \forall t,s \in [0,1]. \tag{3}
$$

Consider the cone

$$
C = \{x \in X : x(t) \ge 0,\ t \in [0,1]\}.
$$

It is clear that (H1) and (H2) imply (A2) and (A3).

Let

$$
\begin{aligned}
\Omega_1 =& \left\{x \in X : \delta\|x\| < |x(t)| < b,\ t \in [0,1]\right\}, \\
\Omega_2 =& \{x \in X : \|x\| < B\}.
\end{aligned}
$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$
\overline{\Omega}_1 = \left\{x \in X : \delta\|x\| \le |x(t)| \le b,\ t \in [0,1]\right\} \subset \Omega_2.
$$

Moreover, $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \ne \emptyset$. Let $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$, then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that (A4) holds.

Next, we will show (A5) holds. Suppose that there exist $u_0 \in \partial\Omega_2 \cap C \cap \mathrm{dom}L$ and $\lambda_0 \in (0,1)$ such that $Lu_0 \in \lambda_0 Nu_0$, then $D^{\alpha}_{0+}u_0(t) \in \lambda_0 f(t,u_0(t))$ for all $t \in [0,1]$. In view of (H3), we get that there exists $w^* \in f(t,u_0(t))$ such that

$$
\begin{aligned}
D^{\alpha}_{0+}u_0(t) =&\ \lambda_0 w^* \le -\lambda_0 b_1|w^*| + \lambda_0 b_2 u_0(t) + \lambda_0 b_3 \\
=&\ -b_1\left|D^{\alpha}_{0+}u_0(t)\right| + \lambda_0 b_2 u_0(t) + \lambda_0 b_3 \\
\le&\ -b_1\left|D^{\alpha}_{0+}u_0(t)\right| + b_2 u_0(t) + b_3,
\end{aligned} \tag{4}
$$

and

$$D_{0+}^{\alpha}u_0(t) = \lambda_0 w^* \le -\lambda_0 c_1 u_0(t) + \lambda_0 c_2, \tag{5}$$

From (4), we obtain

$$\begin{aligned} 0 = u_0(0) - u_0(1) &= \left(I_{0+}^{\alpha} D_{0+}^{\alpha} u_0\right)(1) \\ &\le -\frac{b_1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left|D_{0+}^{\alpha} u_0(s)\right| ds + \frac{b_2}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} u_0(s) ds \\ &\quad + \frac{b_3}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds, \end{aligned}$$

which gives

$$\int_0^1 (1-s)^{\alpha-1} \left|D_{0+}^{\alpha} u_0(s)\right| ds \le \frac{b_2}{b_1} \int_0^1 (1-s)^{\alpha-1} u_0(s) ds + \frac{b_3}{\alpha b_1}.$$

From (5), we obtain

$$\int_0^1 (1-s)^{\alpha-1} u_0(s) ds \le \frac{c_2}{\alpha c_1}.$$

From (3) and the equation

$$u_0 = (I-P)u_0 + Pu_0 = K_P L(I-P)u_0 + Pu_0 = Pu_0 + K_P L u_0,$$

we can get

$$\begin{aligned} u_0 &= \alpha \int_0^1 (1-s)^{\alpha-1} u_0(s) ds + \int_0^1 k(t,s) D_{0+}^{\alpha} u_0(s) ds \\ &\le \frac{c_2}{c_1} + \int_0^1 |k(t,s)| \cdot \left|D_{0+}^{\alpha} u_0(s)\right| ds \\ &= \frac{c_2}{c_1} + \int_0^1 \frac{|k(t,s)|}{(1-s)^{\alpha-1}} \cdot (1-s)^{\alpha-1} \left|D_{0+}^{\alpha} u_0(s)\right| ds \\ &\le \frac{c_2}{c_1} + 3 \int_0^1 (1-s)^{\alpha-1} \left|D_{0+}^{\alpha} u_0(s)\right| ds \\ &\le \frac{c_2}{c_1} + 3 \left[\frac{b_2}{b_1} \int_0^1 (1-s)^{\alpha-1} u_0(s) ds + \frac{b_3}{\alpha b_1}\right] \\ &\le \frac{c_2}{c_1} + \frac{3b_2 c_2}{\alpha b_1 c_1} + \frac{3b_3}{\alpha b_1}. \end{aligned}$$

Then, we have

$$B = \|u_0\| \le \frac{c_2}{c_1} + \frac{3b_2 c_2}{\alpha b_1 c_1} + \frac{3b_3}{\alpha b_1},$$

which contradicts (H3). Hence (A5) holds.

To prove (A6), consider $x \in \mathrm{Ker}L \cap \overline{\Omega}_2$, then $x(t) \equiv c$ on [0, 1]. Let

$$\begin{aligned} H(c,\lambda) &= [I-\lambda(P+JQN)\gamma]c \\ &= c-\lambda\alpha\int_0^1 (1-s)^{\alpha-1}|c|ds-\lambda\alpha\int_0^1 (1-s)^{\alpha-1}f(s,|c|)ds \\ &= c-\lambda|c|-\lambda\alpha\int_0^1 (1-s)^{\alpha-1}f(s,|c|)ds, \end{aligned}$$

for $c \in [-B,B]$ and $\lambda \in [0,1]$. It is easy to show that $0 \in H(c,\lambda)$ implies $c \geq 0$. Suppose $0 \in H(B,\lambda)$ for some $\lambda \in (0,1]$. Then,

$$0 = B-\lambda B-\lambda\alpha\int_0^1 (1-s)^{\alpha-1}w(s,B)ds,$$

where $w \in f(t,B),\ t \in [0,1]$. So (H3) leads to

$$0 \leq B(1-\lambda) = \lambda\alpha\int_0^1 (1-s)^{\alpha-1}w(s,B)ds \leq \lambda(-c_1B+c_2) < 0,$$

which is a contradiction. In addition, if $\lambda = 0$, then $B = 0$, which is impossible. Thus, $H(x,\lambda) \neq 0$ for $x \in \mathrm{Ker}L \cap \partial\Omega_2$, $\lambda \in [0,1]$. As a result,

$$\begin{aligned} &\deg\{[I-(P+JQN)\gamma]_{\mathrm{Ker}L}, \mathrm{Ker}L \cap \Omega_2, 0\} \\ &\quad = \deg\{H(\cdot,1), \mathrm{Ker}L \cap \Omega_2, 0\} \\ &\quad = \deg\{H(\cdot,0), \mathrm{Ker}L \cap \Omega_2, 0\} \\ &\quad = \deg\{I, \mathrm{Ker}L \cap \Omega_2, 0\} = 1 \neq 0. \end{aligned}$$

So (A6) holds.

Next, we prove (A7). Letting $u_0(t) \equiv 1$, so we have $u_0 \in C \setminus \{0\}$ and $C(u_0) = \{x \in C : x(t) > 0, t \in [0,1]\}$. We can take $\sigma(u_0) = 1$. For $x \in C(u_0) \cap \partial\Omega_1$, we get $x(t) > 0, 0 < \|x\| \leq b$ and $x(t) \geq \delta\|x\|,\ t \in [0,1]$.

By (H3) and (H4), for every $x \in C(u_0) \cap \partial\Omega_1$ and $v \in \Psi x$, there exits $w \in Nx$ such that

$$\begin{aligned} v(t_0) &= \alpha\int_0^1 (1-s)^{\alpha-1}x(s)ds+\int_0^1 G(t_0,s)(1-s)^{\alpha-1}w(s,x(s))ds \\ &\geq \delta\|x\|+\int_0^1 G(t_0,s)(1-s)^{\alpha-1}q(s)h(x(s))ds \\ &= \delta\|x\|+\int_0^1 G(t_0,s)(1-s)^{\alpha-1}q(s)\cdot\frac{h(x(s))}{x^\rho(s)}x^\rho(s)ds \\ &\geq \delta\|x\|+\delta^\rho\|x\|^\rho\int_0^1 G(t_0,s)(1-s)^{\alpha-1}q(s)\cdot\frac{h(b)}{b^\rho}ds \\ &= \delta\|x\|+\delta^\rho\|x\|\cdot\frac{b^{1-\rho}}{\|x\|^{1-\rho}}\int_0^1 G(t_0,s)(1-s)^{\alpha-1}q(s)\frac{h(b)}{b}ds \\ &\geq \delta\|x\|+\delta^\rho\|x\|\cdot\int_0^1 G(t_0,s)(1-s)^{\alpha-1}q(s)\frac{h(b)}{b}ds \\ &\geq \delta\|x\|+\delta^\rho\|x\|\cdot\frac{1-\delta}{\delta^\rho} \\ &= \|x\|. \end{aligned}$$

Thus, $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for all $x \in C(u_0) \cap \partial\Omega_1$, i.e., (A7) holds.

Since for $x \in \partial\Omega_2$ and $w \in N\gamma x$, from (H2) we have

$$\begin{aligned}(P+JQN)\gamma x &= \alpha \int_0^1 (1-s)^{\alpha-1}|x(s)|ds + \alpha \int_0^1 (1-s)^{\alpha-1} w(s,|x(s)|)ds \\ &\geq \alpha \int_0^1 (1-s)^{\alpha-1}(1-\kappa)|x(s)|ds \\ &\geq 0.\end{aligned}$$

Thus, $(P+JQN)\gamma x \subset C$ for $x \in \partial\Omega_2$. Then (A8) holds.

Next, we prove (A9). Let $x \in \overline{\Omega}_2 \backslash \Omega_1$

$$\begin{aligned}\Psi_\gamma x(t) = \Bigg\{ & v \in X : \exists w \in N_\gamma x \text{ such that } v = \alpha \int_0^1 (1-s)^{\alpha-1}|x|ds \\ & + \int_0^1 G(t,s)(1-s)^{\alpha-1} w(s,|x|)ds \Bigg\}.\end{aligned}$$

According to (H3) and (2), for $x \in \overline{\Omega}_2 \backslash \Omega_1$ and $v \in \Psi_\gamma x$, there exits $w \in N_\gamma x$ such that

$$\begin{aligned}v(t) &= \alpha \int_0^1 (1-s)^{\alpha-1}|x(s)|ds + \int_0^1 G(t,s)(1-s)^{\alpha-1} w(s,|x(s)|)ds \\ &> \int_0^1 (1-s)^{\alpha-1}|x(s)|(1-\kappa G(t,s))ds \\ &\geq 0.\end{aligned}$$

Hence, $\Psi_\gamma\left(\overline{\Omega}_2 \backslash \Omega_1\right) \subset C$; i.e., (A9) holds.

To prove (A10), suppose there exists $u_0 \in \partial\Omega_2 \cap \mathrm{Ker}L$, i.e., $u_0 = c \in \mathbb{R}$ and $|c| = B$ such that $c \in (P+JQN)\gamma u$. For $w \in N\gamma c$, we have

$$\begin{aligned}c &= \alpha \int_0^1 (1-s)^{\alpha-1}|c|ds + \alpha \int_0^1 (1-s)^{\alpha-1} w(s,|c|)ds \\ &\geq \alpha \int_0^1 (1-s)^{\alpha-1}(1-\kappa)|c|ds \\ &\geq 0.\end{aligned}$$

Hence, we get $c \in (P+JQN)\gamma u$ implies $c \geq 0$. Then for $c = B$ and $w \in N\gamma B$, we have

$$B = \alpha \int_0^1 (1-s)^{\alpha-1} B ds + \alpha \int_0^1 (1-s)^{\alpha-1} w(s,B)ds.$$

Hence,

$$\alpha \int_0^1 (1-s)^{\alpha-1} w(s,B)ds = 0.$$

On the other hand, from (H3), we have

$$0 = \alpha \int_0^1 (1-s)^{\alpha-1} w(s,B)ds \leq -c_1 B + c_2 < 0.$$

This contradiction implies (A10) holds.

Hence, applying Theorem 1, BVP (1) has a positive solution u^* on [0, 1] with $b \le \|u^*\| \le B$. This completes the proof. □

Example

To illustrate how our main result can be used in practice, we present here an example.

Let us consider the following fractional differential inclusion at resonance

$$\begin{cases} D_{0^+}^{1.5}u(t) \in f(t,u), \quad 0 \le t \le 1, \\ u'(0) = 0, \quad u(0) = u(1), \end{cases} \tag{6}$$

where $f(t,u) = \left\{w(t,u) + \frac{1}{25}v : v \in [0,1]\right\}, w(t,u) = \frac{1}{300}\left(1+2t-2t^2\right)\left(u^2-4u+3\right)u.$

Corresponding to BVP (1), we have that $\alpha = 1.5$ and

$$G(t,s) = \begin{cases} \frac{3}{2} - \frac{\Gamma(2.5)}{\Gamma(3)}(1-s)^{1.5} - \frac{1.5t^{1.5}}{\Gamma(2.5)} + \frac{1.5\Gamma(2.5)}{\Gamma(3)}, \quad 0 \le t < s \le 1, \\ \frac{3}{2} - \frac{\Gamma(2.5)}{\Gamma(3)}(1-s)^{1.5} - \frac{1.5t^{1.5}}{\Gamma(2.5)} + \frac{1.5\Gamma(2.5)}{\Gamma(3)} + \frac{1}{\Gamma(1.5)}\left(\frac{t-s}{1-s}\right)^{0.5}, \quad 0 \le s < t \le 1, \end{cases}$$

It is easy to see that $G(t,s) \ge 0$ for $t,s \in [0,1]$.

Let $\kappa = 0.003$ and $B = 2$. By the monotonicity of the function, for $x \in [0,2]$ and $w \in f(t,x)$, $t \in [0,1]$, we can prove that

$$-\frac{3}{1000}x \le w(t,x) \le -\frac{1}{30}x + \frac{1}{17}$$

and

$$w(t,x) \le -\frac{8}{3}|w| + \frac{1}{30}x + \frac{1}{4}.$$

Then, we can choose $c_1 = \frac{1}{30}, c_2 = \frac{1}{17}, b_1 = \frac{8}{3}, b_2 = \frac{1}{30}, b_3 = \frac{1}{4}$. By calculation, we have

$$\frac{c_2}{c_1} + \frac{3b_2c_2}{\alpha b_1 c_1} + \frac{3b_3}{\alpha b_1} \approx 1.764 + 0.044 + 0.187 = 1.808 < 2 = B.$$

Take $q(t) = \frac{1}{240}\left(1+2t-t^2\right)$ and $h(x) = x$. We see that $q \in L^1[0,1]$, $q(t) \ge 0$ and $h \in C\left((0,b],\mathbb{R}^+\right)$, where $b = 1/2 \in (0,B) = (0,2)$. Furthermore, for $(t,u) \in [0,1] \times (0,1/2]$ and $w \in f(t,u)$, by a simple computation, we get that

$$w(t,u) \ge q(t)h(u).$$

Choose $\rho = 1$, so we have $\frac{h(u)}{u^\rho} \equiv 1$ which is non-increasing on $(0,b]$. By Choosing $t_0 = 0$, $\delta = 0.997$, with simple calculations, we can get

$$\frac{h(b)}{b}\int_0^1 G(t_0,s)(1-s)^{\alpha-1}q(s)ds \ge \frac{1-\delta}{\delta^\rho}.$$

Therefore, (H_1)–(H_4) of Theorem 2 are satisfied. Then BVP (6) has a positive solution on [0, 1].

Conclusions

In this paper, we have obtained the existence of positive solutions for a boundary value problem of fractional differential inclusions at resonance. By using the Leggett–Williams theorem for coincidences of multi-valued operators due to O'Regan and Zima, we have found the existence results. Our results are new in the context of fractional differential inclusions and positive solutions. As applications, an example is presented to illustrate the main results. In the future, we will consider the the uniqueness of positive solutions for the fractional differential equations at resonance.

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Competing interests
The author declares that he has no competing interests.

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References

Agarwal RP, Benchohra M, Hamani S (2010) A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl Math 109:973–1033
Bai Z (2011) Solvability for a class of fractional m-point boundary value problem at resonance. Comput Math Appl 62:1292–1302
Bai Z, Lü H (2005) Positive solutions for boundary value problem of nonlinear fractional differential equation. J Math Anal Appl 311:495–505
Bai Z, Zhang Y (2011) Solvability of fractional three-point boundary value problems with nonlinear growth. Appl Math Comput 218:1719–1725
Caballero J, Harjani J, Sadarangani K (2011) Positive solutions for a class of singular fractional boundary value problems. Comput Math Appl 63:1325–1332
Chen Y, Tang X, He X (2013) Positive solutions of fractional differential inclusion at resonace. Mediterr J Math 10:1207–1220
Goodrich CS (2010) Existence of a positive solution to a class of fractional differential equation. Appl Math Lett 23:1050–1055
Kilbas AA, Srivastava HM, Trujillo JJ (2006) Theory and applications of fractional differential equations. North-Holland mathematics studies, vol 204. Elsevier Science B.V., Amsterdam
Kosmatov N (2008) Multi-point boundary value problems on an unbounded domain at resonance. Nonlinear Anal 68:2158–2171
Kosmatov N (2010) A boundary value problem of fractional order at resonance. Electron J Differ Equ 135:1–10
Lin W (2007) Global existence theory and chaos control of fractional differential equations. J Math Anal Appl 332:709–726
Mawhin J (1979) Topological degree methods in nonlinear boundary value problems. NSFCBMS regional conference series in mathematics. American Mathematical Society, Providence
O'Regan D, Zima M (2008) Leggett–Williams theorems for coincidences of multivalued operators. Nonlinear Anal 68:2879–2888
Orsingher E, Beghin L (2004) Time-fractional telegraph equations and telegraph processes with brownian time. Probab Theory Relat Fields 128:141–160
Podlubny I (1999) Fractional differential equations. Academic Press, New York
Samko SG, Kilbas AA, Marichev OI (1993) Fractional integrals and derivatives. Theory and applications. Gordon and Breach, Yverdon
Xu X, Jiang D, Yuan C (2009) Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation. Nonlinear Anal 71:4676–4688
Yang A, Wang H (2011) Positive solutions for two-point boundary value problems of nonlinear fractional differential equation at resonace. Electron J Qual Theory Differ Equ 71:1–15
Zhang S (2006) Positive solutions for boundary-value problems of nonlinear fractional differential equations. Electron J Differ Equ 36:1–12