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Ideals and primitive elements of some relatively free Lie algebras

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Abstract

Let F be a free Lie algebra of finite rank over a field K. We prove that if an ideal $\langle \tilde{v} \rangle$ of the algebra $F/\gamma_{m+1}(F')$ contains a primitive element $\tilde{u}$ then the element $\tilde{v}$ is primitive. We also show that, in the Lie algebra $F/\gamma_3(F)'$ there exists an element $\bar{v}$ such that the ideal $\langle \bar{v} \rangle$ contains a primitive element $\bar{u}$ but, $\bar{u}$ and $\bar{v}$ are not conjugate by means of an inner automorphism.

Keywords: Primitive element, Free Lie algebra, Ideal, Lower central series, Free nilpotent

Background

Let F be a free Lie algebra of finite rank n, with $n \geq 2$, freely generated by the set $\{x_1, \ldots, x_n\}$ over a field K. By F' and F'' we denote the subalgebras $[F, F]$ and $[F', F']$ of F respectively. An ideal V in the free Lie algebra F is called a verbal ideal if for any $g(x_1, \ldots, x_n) \in V$ and any $h_1, \ldots, h_n \in F$ the Lie polynomial $g(h_1, \ldots, h_n)$ belongs to V. Let V be a non-trivial verbal ideal of F. An element of F is said to be primitive if it can be included in a free generating set of F. Similarly an element of the relatively free Lie algebra F/V is called primitive if it is extendible to a free generating set of F/V.

Let $L = F/F''$ be the free metabelian Lie algebra. Write $\bar{x}_i = x_i + F'', i = 1, 2, \ldots, n$. Thus, the set $\{\bar{x}_1, \ldots, \bar{x}_n\}$ is a free generating set for L (Bahturin 1987). For $g \in L$, let $\langle g \rangle$ be the ideal generated by g and let h be a primitive element of L. It is known that if $h \in \langle g \rangle$ then g is a primitive element in L (Chirkov and Shevelin 2001). In fact there is an inner automorphism θ of L such that $\theta(h) = g$. For each $v \in L'$ the linear operator

$$adv : L \longrightarrow L$$

defined by

$$adv(w) = [w, v], \quad w \in L$$

is a derivation of L and $ad^2 v = 0$ because $L'' = \{0\}$. Hence the linear mapping

$$\exp(adv) = 1 + adv$$

is well defined and it is an inner automorphism of L. In Chirkov and Shevelin (2001) proved that for $g \in L$ if a primitive element h of L belongs to the ideal $\langle g \rangle$ then h and g are conjugate by means of an inner automorphism of L. This result was obtained by

Evans (1994) for free metabelian groups. Does a similar result, as in L, holds for the Lie algebras $F/\gamma_{m+1}(F')$ and $F/\gamma_3(F)'$? In the group case this question was answered by Timoshenko (1997). In the present paper we answer this question. We obtain an affirmative answer for the Lie algebra $F/\gamma_{m+1}(F')$. In contrast to the case of free metabelian Lie algebras and free Lie algebras of the form $F/\gamma_{m+1}(F')$, for the Lie algebra $F/\gamma_3(F)'$ we prove that the question has a negative answer. Our main results are similar to the result of Timoshenko (1997) in the case of groups but there are some essential differences.

Preliminaries

Let F be the free Lie algebra generated by a set $X = \{x_1, \ldots, x_n\}$ over a field K of characterisitic zero, $U(F)$ be the universal enveloping algebra of F and Δ its augmentation ideal, that is, the kernel of the natural homomorphism $\sigma : U(F) \longrightarrow K$ defined by $\sigma(x_i) = 0, 1 \leq i \leq n$. For a given subalgebra R of F we denote by Δ_R the left ideal of $U(F)$ generated by the subalgebra R. In the case where R is an ideal of F, Δ_R becomes a two-sided ideal of $U(F)$. In fact Δ_R is the kernel of the natural homomorphism $U(F) \longrightarrow U(F/R)$. For any element u of F we denote by $\langle u \rangle$ the ideal of F generated by the element u.

Fox (1953) gave a detailed account of the differential calculus in a free group ring. We introduce here free derivations $\frac{\partial}{\partial x_i} : U(F) \longrightarrow U(F), 1 \leq i \leq n$ such that $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$(Kronecker delta), $\frac{\partial (uv)}{\partial x_i} = \frac{\partial u}{\partial x_i}\sigma(v) + u\frac{\partial v}{\partial x_i}$. It is an obvious consequence of the definitions that $\frac{\partial}{\partial x_i}(1) = 0$. The ideal Δ is a free left $U(F)$-module with a free basis X and the mappings $\frac{\partial}{\partial x_i}$ are projections to the corresponding free cyclic direct summands. Thus any element $f \in \Delta$ can be uniquely written in the form

$$f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i.$$

For any elements $g_1, \ldots, g_n$ of $U(F)$ we can always find an element f of $U(F)$ such that $\frac{\partial f}{\partial x_i} = g_i, 1 \leq i \leq n$.

Let ∂f be the column vector $\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)^T$, where T indicates transpose.

For any Lie algebra G, the lower central series

$$G = \gamma_1(G) \supseteq \cdots \supseteq \gamma_k(G) \supseteq \cdots$$

is defined inductively by $\gamma_2(G) = [G, G], \gamma_k(G) = [\gamma_{k-1}(G), G], k \geqslant 2$. We usually write G', for $\gamma_2(G)$.

Let R be an ideal of F. If u is an element of F, then we denote the images of u under the natural homomorphisms as follows: by $\widehat{u}$ in F / R, by $\overline{u}$ in F/R' and by $\widetilde{u}$ in $F/\gamma_{m+1}(R)$, where $m \geq 1$.

In (Umirbaev 1993), Umirbaev has defined the right derivatives in the algebras F/R' and $F/\gamma_{m+1}(R)$. We give a summary here referring to (Umirbaev 1993).

Let

$$\rho : \left[U(F)^n \right]^T \to \left[U(F/R)^n \right]^T,$$

be the natural componentwise homomorphism, i.e.,

$$\rho\left(\left(f_1,\ldots,f_n\right)^T\right) = \left(\widehat{f_1},\ldots,\widehat{f_n}\right)^T.$$

where $\left(\widehat{f_1},\ldots,\widehat{f_n}\right)^T$ is the transpose of the vector $\left(\widehat{f_1},\ldots,\widehat{f_n}\right)$.

Consider the composition mapping

$$\rho \circ \partial : F \overset{\partial}{\to} \left[U(F)^n\right]^T \overset{\rho}{\to} \left[U(F/R)^n\right]^T. \tag{1}$$

This mapping induces the mappings

$$\overline{\partial} : F/R' \to \left[U(F/R)^n\right]^T, \quad \widetilde{\partial} : F/\gamma_{m+1}(R) \to \left[U(F/R)^n\right]^T.$$

Since the kernel of the mapping $\widetilde{\partial}$ is $R'/\gamma_{m+1}(R)$ (see Umirbaev 1993 for details) then it induces the mapping $\overline{\overline{\partial}} : H \to [U(F/R)^n]^T$, where $H = F/\gamma_{m+1}(R)/R'/\gamma_{m+1}(R)$.

For any element f of F the components $\frac{\overline{\partial}\,\overline{f}}{\partial x_i}$, $\frac{\widetilde{\partial}\,\widetilde{f}}{\partial x_i}$ and $\frac{\overline{\overline{\partial}}\,\overline{\overline{f}}}{\partial x_i}$ of the vectors

$$\overline{\partial}\left(\overline{f}\right) = \left(\frac{\overline{\partial}\,\overline{f}}{\partial x_1},\ldots,\frac{\overline{\partial}\,\overline{f}}{\partial x_n}\right)^T, \widetilde{\partial}\left(\widetilde{f}\right) = \left(\frac{\widetilde{\partial f}}{\partial x_1},\ldots,\frac{\widetilde{\partial f}}{\partial x_n}\right)^T, \overline{\overline{\partial}} = \left(\frac{\overline{\overline{\partial}}\,\overline{\overline{f}}}{\partial x_1},\ldots,\frac{\overline{\overline{\partial}}\,\overline{\overline{f}}}{\partial x_n}\right)$$

are called the partial derivatives of $\overline{f}$, $\widetilde{f}$ and $\overline{\overline{f}}$ respectively. Here we use left derivatives instead of right derivatives.

For each $u \in R/\gamma_{m+1}(R)$ the derivation $adu : F/\gamma_{m+1}(R) \longrightarrow F/\gamma_{m+1}(R)$ is nilpotent and $(adu) = 0$, because $\gamma_{m+1}(R/\gamma_{m+1}(R)) = \{0\}$.

Hence the linear mapping

$$\exp(adu) = 1 + \frac{adu}{1!} + \frac{ad^2u}{2!} + \cdots + \frac{ad^mu}{m!}$$

is well defined and it is an inner automorphism of $F/\gamma_{m+1}(R), m \geq 1$, that is, since $[[w, \underbrace{u],\ldots,u}_{(m+1)-times}] = 0,$

$$\exp(adv)(w) = w + \frac{[w,u]}{1!} + \frac{[[w,u],u]}{2!} + \cdots + \frac{\overbrace{[[w,u],\ldots,u]}^{m-times}}{m!}.$$

We need the following technical lemmas. The first lemma is an immediate consequence of the definitions.

Lemma 1 *Let J be an arbitrary ideal of $U(F)$ and $u \in \Delta$. Then $u \in J\Delta$ if and only if $\frac{\partial u}{\partial x_i} \in J$ for each $i, 1 \leq i \leq n$.*

The next lemma can be found in Yunus (1984).

Lemma 2 *Let R be an ideal of F and $u \in F$. Then $u \in \Delta_R\Delta$ if and only if $u \in R'$.*

Main results

Let F be the free Lie algebra generated by a set $X = \{x_1, \ldots, x_n\}, n \geq 2$, over a field K of characteristic zero and let R be a non-trivial verbal ideal of F.

For an element f of F the vector $\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ is called unimodular, if there exist $a_1, \ldots, a_n \in U(F)$ such that

$$a_1 \frac{\partial f}{\partial x_1} + \cdots + a_n \frac{\partial f}{\partial x_n} = 1.$$

Umirbaev (1993) has proved a criterion of primitiveness for a system of elements in a finitely generated free Lie algebra of the form $F/\gamma_{m+1}(R)$, where $m \geq 1$ and $R = F'$. Umirbaev's criterion for the primitivity of an element of the algebra $F/\gamma_{m+1}(R)$ is stated below.

Proposition 3 *Let $R = F'$. An element $\widetilde{u}$ of $F/\gamma_{m+1}(R)$ is primitive if and only if the vector $\left(\frac{\partial \widetilde{u}}{\partial x_1}, \ldots, \frac{\partial \widetilde{u}}{\partial x_n}\right)$ is unimodular in $U(F/R)$.*

We are going to consider the case $R = F'$.

Proposition 4 *An element $\overline{f}$ of the free metabelian Lie algebra F/F'' is primitive if and only if the image $\widetilde{f}$ is primitive in the free nilpotent-by-abelian Lie algebra $F/\gamma_{m+1}(F')$, where $f \in F$, $m \geq 2$.*

Proof Suppose that the element $\overline{f}$ of F/F'' is primitive. If we put $m = 1$ in Proposition 3 we have that the vector $\left(\frac{\overline{\partial} \overline{f}}{\partial x_1}, \ldots, \frac{\overline{\partial} \overline{f}}{\partial x_n}\right)$ is unimodular in $U(F/F')$, that is, there exist $a_1, \ldots, a_n \in U(F/F')$ such that $\sum_{i=1}^{n} a_i \frac{\overline{\partial} \overline{f}}{\partial x_i} = 1$.

Let $H = F/\gamma_{m+1}(F')/F''/\gamma_{m+1}(F')$. We calculate the derivative $\frac{\overline{\partial} \overline{f}}{\partial x_i}$ by using the natural homomorphism $\theta : F/\gamma_{m+1}(F') \to H$, the isomorphism $\varphi : H \to F/F''$ and the chain rule for derivatives:

$$\begin{aligned}
\frac{\overline{\partial} \overline{f}}{\partial x_i} &= \frac{\overline{\partial} \varphi\left(\overline{\overline{f}}\right)}{\partial x_i} \\
&= \frac{\overline{\partial} \varphi\left(\overline{\overline{f}}\right)}{\partial \overline{\overline{f}}} \cdot \frac{\overline{\overline{\partial}} \overline{\overline{f}}}{\partial x_i} \\
&= \frac{\overline{\partial} \varphi\left(\overline{\overline{f}}\right)}{\partial \overline{\overline{f}}} \cdot \frac{\overline{\overline{\partial}} \theta\left(\widetilde{f}\right)}{\partial x_i} \\
&= \frac{\overline{\partial} \varphi\left(\overline{\overline{f}}\right)}{\partial \overline{\overline{f}}} \cdot \frac{\overline{\overline{\partial}} \theta\left(\widetilde{f}\right)}{\partial \widetilde{f}} \cdot \frac{\widetilde{\partial} \widetilde{f}}{\partial x_i}.
\end{aligned}$$

It is clear that $\frac{\overline{\partial} \varphi\left(\overline{\overline{f}}\right)}{\partial \overline{\overline{f}}}, \frac{\overline{\overline{\partial}} \theta\left(\widetilde{f}\right)}{\partial \widetilde{f}} \in U(F/F')$. Therefore from the equality $\sum_{i=1}^{n} a_i \frac{\overline{\partial} \overline{f}}{\partial x_i} = 1$ we get $\sum_{i=1}^{n} b_i \frac{\widetilde{\partial} \widetilde{f}}{\partial x_i} = 1$, where $b_i = a_i \frac{\overline{\partial} \varphi\left(\overline{\overline{f}}\right)}{\partial \overline{\overline{f}}} \cdot \frac{\overline{\overline{\partial}} \theta\left(\widetilde{f}\right)}{\partial \widetilde{f}}$. Hence by Proposition 3, $\widetilde{f}$ is primitive in $F/\gamma_{m+1}(F')$.

Now suppose that $\widetilde{f}$ is primitive element of the algebra $F/\gamma_{m+1}(F')$. By definition it can be extended to a free generating set $Y = \left\{ \widetilde{f} = \widetilde{f}_1, \ldots, \widetilde{f}_n \right\}$ of $F/\gamma_{m+1}(F')$. Clearly Y is linearly independent modulo $\left(F/\gamma_{m+1}(F') \right)'$. Therefore the image $\theta(Y)$ of Y in the algebra H is linearly independent modulo H'. As a simple application of theorem 4.2.4.9 of Bahturin (1987) we see that $\theta(Y)$ freely generates the algebra H. Hence the image of $\theta(Y)$ under the isomorphism $\varphi : H \to F/F''$ generates the algebra F/F''. That is, the algebra F/F'' is freely generated by the set $\varphi(\theta(Y)) = \left\{ \overline{f} = \overline{f}_1, \ldots, \overline{f}_n \right\}$. Thus, $\overline{f}$ is a primitive element of the algebra F/F''. ☐

As a consequence of the result of Chirkov and Shevelin (2001), we obtain the following proposition. Although its proof is given in Ersalan and Esmerligil (2014), our proof is more explicit. The idea of the proof is similar to the idea of the proof of Proposition 2 of the paper by Timoshenko (1997) for groups.

Proposition 5 *Let $\widetilde{u}$ be a primitive element of the algebra $F/\gamma_{m+1}(F')$ and let $\widetilde{v} \in F/\gamma_{m+1}(F')$ where $u, v \in F, m \geq 1$. If $\widetilde{u} \in \langle \widetilde{v} \rangle$ then $\widetilde{v}$ is also primitive.*

Proof Let $\widetilde{u}, \widetilde{v} \in F/\gamma_{m+1}(F')$. Assume that $\widetilde{u}$ is primitive and that it is contained in the ideal $\langle \widetilde{v} \rangle$ of $F/\gamma_{m+1}(F')$. By Proposition 4, $\overline{u}$ is primitive. In the view of Proposition 4, it sufficies to the prove that the element $\overline{v}$ of F/F'' is primitive.

Since

$$\widetilde{u} = u + \gamma_{m+1}(F') \in \langle v + \gamma_{m+1}(F') \rangle, \quad m \geq 1$$

we have

$$u \in \langle v \rangle \left(mod F'' \right),$$

that is,

$$\overline{u} = u + F'' \in \langle v + F'' \rangle.$$

From the result of Chirkov and Shevelin (2001), we obtain that the elements $\overline{u}$ and $\overline{v}$ are conjugate by means of an inner automorphism. Therefore $\overline{v}$ is primitive. Hence the result follows. ☐

The mapping $\widehat{} : F \to F/R$ can be extended to the mapping $\widehat{} : U(F) \to U(F/R)$ for which we preserve the same notation.

The following lemma will play a crucial role in proving our main result.

Lemma 6 *Let R be a verbal ideal of $F, r \in R$ and let $v \in F$. Then $r + R' \in \langle v \rangle + R'$ if and only if there exist an element $\widehat{\alpha} \in U(F/R)$ and an element $\widehat{\beta}_i \in \Delta_{\widehat{v}}$, such that $\frac{\widehat{\partial r}}{\partial x_i} = \widehat{\alpha} \frac{\widehat{\partial v}}{\partial x_i} + \widehat{\beta}_i$, where $i = 1, \ldots, n$ and $\Delta_{\widehat{v}}$ is the ideal generated by the element $\widehat{v}$ in the algebra $U(F/R)$.*

Proof Let r be an element of the ideal $R, v \in F$ and $\overline{r} \in \langle \overline{v} \rangle$. Then $r \in \langle v \rangle \left(mod R' \right)$, where $\langle v \rangle$ is the ideal of F generated by v. Any element of the ideal $\langle v \rangle$ can be written as linear combinations of commutators of F depending on the element v. Applying the Jacobi

identitiy and the anticommutativity, these commutators can be rewritten as linear combinations of commutators of the form

$$[[\dots [[v, x_{i_1}], x_{i_2}], \dots], x_{i_k}], \quad x_{i_1}, \dots x_{i_k} \in \{x_1, \dots, x_n\}, \quad k \geq 0. \tag{2}$$

If $r \equiv v \, (mod R')$ then clearly $\frac{\widehat{\partial r}}{\partial x_i} = \frac{\widehat{\partial v}}{\partial x_i}, i = 1, \dots n.$

Now assume that the element r is written as a linear combination of elements of the form (2). Without loss of generality we may assume that

$$r \equiv [[\dots [v, x_{i_1}], \dots], x_{i_k}] \, (mod R'), \quad k \geq 1,$$

By straightforward calculations we see that the form of the derivatives $\frac{\partial r}{\partial x_i}$ are

$$\frac{\partial r}{\partial x_i} = \alpha \frac{\partial v}{\partial x_i} + \beta_i + \Delta_R,$$

where $\alpha \in U(F/R), \beta_i \in \Delta_v, i = 1, \dots, n.$

Therefore

$$\frac{\widehat{\partial r}}{\partial x_i} = \widehat{\alpha} \frac{\widehat{\partial v}}{\partial x_i} + \widehat{\beta_i}.$$

Let now $\frac{\widehat{\partial r}}{\partial x_i} = \widehat{\alpha} \frac{\widehat{\partial v}}{\partial x_i} + \widehat{\beta_i}$, where $\widehat{\alpha} \in U(F/R), \widehat{\beta_i} \in \Delta_{\widehat{v}}, i = 1, \dots n.$

The kernel of the natural homomorphism $\widehat{\ } : U(F) \longrightarrow U(F/R)$ is Δ_R, and hence

$$\frac{\partial r}{\partial x_i} + \Delta_R = \alpha \frac{\partial v}{\partial x_i} + \beta_i + \Delta_R.$$

Then there exists an element g of Δ_R such $\frac{\partial}{\partial x_i}(r - \alpha v) = \beta_i + g$, where $\beta_i \in \Delta_v$, that is, $\frac{\partial}{\partial x_i}(r - \alpha v) \in \Delta_v + \Delta_R$. By Lemma 1 we have $r - \alpha v \in \Delta_v \Delta + \Delta_R \Delta$. Hence the element $r - \alpha v$ of F can be written as $r - \alpha v = h + z$, where $h \in \Delta_v \Delta, z \in \Delta_R \Delta$. By Lemma 2 we get $h \in \langle v \rangle'$ and $z \in R'$. Hence $r + R' = \alpha v + h + R'$. This completes the proof. $\qquad\square$

In contrast to the case of free metabelian Lie algebras we can show that there exists an element $\bar{v}$ of the algebra $F/\gamma_3(F)'$ such that the ideal $\langle \bar{v} \rangle$ of $F/\gamma_3(F)'$ contains a primitive element $\bar{u}$, but $\bar{u}$ and $\bar{v}$ are not conjugate by means of an inner automorphism.

Theorem 7 *There is an element $\bar{v}$ in the algebra $F/\gamma_3(F)'$ such that the ideal $\langle \bar{v} \rangle$ of $F/\gamma_3(F)'$ contains the element $\overline{x_1}$, but the elements v and x_1 are not conjugate modulo $\gamma_3(F)'$ by means of an inner automorphism.*

Proof We consider the element $\bar{v} = x_1 + [[[[x_1, x_2], x_2], x_1], x_2] + \gamma_3(F)'$ of $F/\gamma_3(F)'$ which is an analogue of the element given in Fox (1953) for groups. Let $w = [[[[x_1, x_2], x_2], x_1], x_2].$

We have

$$\frac{\partial w}{\partial x_1} = -x_2 \cdot [[x_1, x_2], x_2] + x_2 \cdot x_1 \cdot \frac{\partial [[x_1, x_2], x_2]}{\partial x_1},$$

$$\frac{\partial w}{\partial x_2} = [[[x_1, x_2], x_2], x_1] + x_2 \cdot x_1 \cdot \frac{\partial [[x_1, x_2], x_2]}{\partial x_2}.$$

Now consider the images $\widehat{\frac{\partial w}{\partial x_i}}$ under the homomorphism

$$\widehat{}: U(F) \longrightarrow U(F/\gamma_3(F)), \quad i = 1, 2.$$

Then

$$\widehat{\frac{\partial w}{\partial x_1}} = \widehat{x_2} \cdot \widehat{x_1} \cdot \frac{\partial \widehat{[[x_1, x_2], x_2]}}{\partial x_1},$$

$$\widehat{\frac{\partial w}{\partial x_2}} = \widehat{x_2} \cdot \widehat{x_1} \cdot \frac{\partial \widehat{[[x_1, x_2], x_2]}}{\partial x_2},$$

$$\widehat{\frac{\partial w}{\partial x_k}} = 0, \quad k > 2.$$

Clearly $\widehat{\frac{\partial w}{\partial x_i}} \in \Delta_{\widehat{x_1}} = \Delta_{\widehat{v}}$. In the above equalities if we set $\widehat{\alpha} = 0$ and $\widehat{\beta_i} = \widehat{x_2} \cdot \widehat{x_1} \cdot \frac{\partial \widehat{[[x_1, x_2], x_2]}}{\partial x_i}, i = 1, 2$, then we see that

$$\widehat{\frac{\partial w}{\partial x_i}} = \widehat{\alpha} \cdot \widehat{\frac{\partial v}{\partial x_i}} + \widehat{\beta_i}, \quad i = 1, 2.$$

By Lemma 6 $w + \gamma_3(F)' \in \langle v \rangle + \gamma_3(F)'$. Therefore we have

$$x_1 + \gamma_3(F)' = v - w + \gamma_3(F)' \in \langle v \rangle + \gamma_3(F)'.$$

Now we are going to verify that the element w can not be written in the form $[x_1, u]$ in the algebra $F/\gamma_3(F)'$.

Assume that the rank of F equal to $2, u \in \gamma_3(F)$ and

$$w = [x_1, u]. \tag{3}$$

Let us calculate the derivative $\frac{\partial}{\partial x_1}$ of both sides of (3). We have

$$-x_2 \cdot [[x_1, x_2], x_2] + x_2 \cdot x_1 \cdot x_2 \cdot x_2 = -u + x_1 \cdot \frac{\partial u}{\partial x_1}.$$

Taking the image under the homomorphism $\widehat{}: U(F) \longrightarrow U(F/\gamma_3(F))$ we get

$$\widehat{x_2} \cdot \widehat{x_1} \cdot \widehat{x_2} \cdot \widehat{x_2} = \widehat{x_1} \widehat{\frac{\partial u}{\partial x_1}}. \tag{4}$$

It is well known that the set $\left\{ \widehat{x_1}, \widehat{x_2}, \left[\widehat{x_1}, \widehat{x_2} \right] \right\}$ is a basis of $F/\gamma_3(F)$. Therefore by Poincare–Birkhoff–Witt's theorem the algebra $U(F/\gamma_3(F))$ is a free K- module generated 1 and the all ordered monomials of the form

$$\left[\widehat{x_1}, \widehat{x_2} \right]^r \cdot \widehat{x_1}^s \cdot \widehat{x_2}^k, \quad r \geq 0, s \geq 0, k \geq 0, \quad (r, s, k) \neq (0, 0, 0).$$

Thus every element of $U(F/\gamma_3(F))$ can be uniquely written as

$$\sum_{r,s,k \geq 0} \alpha_{rsk} \left[\widehat{x_1}, \widehat{x_2} \right]^r \cdot \widehat{x_1}^s \cdot \widehat{x_2}^k, \quad \alpha_{ijk} \in K. \tag{5}$$

Let us express each side of (4) in the form (5):

$$\widehat{x_2} \cdot \widehat{x_1} \cdot \widehat{x_2} \cdot \widehat{x_2} = \left[\widehat{x_2}, \widehat{x_1}\right] \widehat{x_2} \cdot \widehat{x_2} + \widehat{x_1} \cdot \widehat{x_2} \cdot \widehat{x_2} \cdot \widehat{x_2},$$

$$\widehat{x_1} \widehat{\frac{\partial u}{\partial x_1}} = \widehat{x_1} \sum_{i,j,k \geq 0} \alpha_{ijk} \left[\widehat{x_1}, \widehat{x_2}\right]^r \cdot \widehat{x_1}^s \cdot \widehat{x_2}^k.$$

Then

$$\widehat{x_1} \cdot \sum_{i,j,k \geq 0} \alpha_{ijk} \left[\widehat{x_1}, \widehat{x_2}\right]^i \cdot \widehat{x_1}^j \cdot \widehat{x_2}^k = \left[\widehat{x_2}, \widehat{x_1}\right] \cdot \widehat{x_2}^2 + \widehat{x_1} \cdot \widehat{x_2}^3. \tag{6}$$

We note that in the algebra $F/\gamma_3(F)$ we have $\left[\left[\widehat{x_1}, \widehat{x_2}\right], \widehat{x_i}\right] = 0$, $i \in \{1, 2\}$. That is, the elements $\left[\widehat{x_1}, \widehat{x_2}\right]$ and $\widehat{x_i}$ commute in the algebra $U(F/\gamma_3(F))$.

So from (6) we get

$$\sum_{i,j \geq 0} \alpha_{ij2} \left[\widehat{x_1}, \widehat{x_2}\right]^i \cdot \widehat{x_1}^{j+1} = \left[\widehat{x_2}, \widehat{x_1}\right] \tag{7}$$

and

$$\sum_{i,j \geq 0} \alpha_{ij3} \left[\widehat{x_1}, \widehat{x_2}\right]^i \cdot \widehat{x_1}^{j+1} = \widehat{x_1}.$$

Using (7) we obtain

$$\sum_{j \geq 0} \alpha_{1j2} \widehat{x_1}^{j+1} = -1,$$

which is impossible. This contradiction proves the theorem. □

Conclusions

In this work we found a relation between the generator of a one-generated ideal of a relatively free Lie algebra and a primitive element which is contained in this ideal. One can expects to adopt our results for ideals of some relatively free Lie algebras which have more than one generator.

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