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Dynamics of the functions $f_\mu(z) = z\exp(z + \mu)$ with the real parameter

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Abstract

In this paper, the dynamics of the functions $f_\mu(z) = z\exp(z + \mu)$ with the real parameter is studied. We say that a real parameter μ belongs to the set B_n for a positive integer n if f_μ has an attracting cycle of n-order. We prove that the Fatou set $F(f_\mu)$ is a completely invariant attracting basin for every parameter $\mu < 0$. Further, regarding the set B_n for $n > 1$, we prove the following results: (1) There exists $\mu_* \neq +\infty$ such that $B_2 = (2, \mu_*)$. (2) For every positive integer $n > 2$, the set B_n is non-empty. (3) For every prime number $p > 3$, the set B_p has at least two components.

Keywords: Julia set, Fatou set, Periodic point, Critical value

Mathematics Subject Classification: Primary 37F10, Secondary 30D05

Introduction and main results

Let f^n be the n-th iterate of a transcendental entire function f. The maximal open set $F(f)$ where the family $\{f^n\}_{n=0}^{\infty}$ is normal in the sense of Montel is called the Fatou set, and its complement $J(f) := \mathbb{C} \backslash F(f)$ is called the Julia set. The dynamics given by the iteration of transcendental entire maps has been widely studied (cf. Eremenko and Lyubich 1992).

Baker (1970) first obtained an entire function f with the property $J(f) = \mathbb{C}$. He proved the following Theorem.

Theorem 1 *For a certain real positive value k, the function $f(z) = kze^z$ has the whole plane for its set $J(f)$.*

After that, many authors (cf. Fagella 1995; Jang 1992; Kuroda and Jang 1997; Morosawa 1998) studied the dynamics of the functions $f_\mu(z) := z\exp(z + \mu)$. Jang (1992) proved that the set

$$B_0 := \{\mu \in \mathbb{R} | J(f_\mu) = \mathbb{C}\}$$

is an infinite set. Further, Morosawa (1998) proved that the one-dimensional Lebesgue measure of B_0 is positive.

The function f_μ has only two singular values: an asymptotic value 0 and a critical value $f_\mu(-1)$, hence the Fatou set $F(f_\mu)$ has no wandering components. The asymptotic value is fixed, hence there is only one free singular orbit. It follows that there is at most one cycle of periodic Fatou components, either attracting, parabolic or Siegel. Since for real parameters the orbit of the free critical value is entirely real, there is no possibility of Siegel discs. Hence only attracting or parabolic cycles are possible and attracting or parabolic periodic points (if they exist) are real.

In this paper, our main goal is to study the structure of B_n, where

$$B_n := \{\mu \in \mathbb{R} | f_\mu \text{ has a cycle of attracting periodic points of } n\text{-order}\},$$

for every positive integer n.

For every real parameter μ, f_μ has two real fixed points 0 and $-\mu$. The multiplier of 0 is e^μ, and the multiplier of $-\mu$ is $1-\mu$. Hence $\mu \in B_1$ if and only if μ satisfies the following condition:

$$-1 < e^\mu < 1 \quad \text{or} \quad -1 < 1-\mu < 1.$$

This immediately implies that $B_1 = (-\infty, 0) \cup (0, 2)$.

Since a completely invariant domain contains all singular values, it is easy to see that if $\mu \in (0, 2)$, then the Fatou set $F(f_\mu)$ is not a completely invariant attracting basin. However, for $\mu \in (-\infty, 0)$, we have the following result.

Theorem 2 *For every parameter $\mu < 0$, the Fatou set $F(f_\mu)$ is a completely invariant attracting basin.*

Regarding the set B_n for $n > 1$, we prove the following Theorems.

Theorem 3 *There exists $\mu_* \neq +\infty$ such that $B_2 = (2, \mu_*)$.*

Theorem 4 *For every positive integer $n > 2$, $B_n \neq \emptyset$.*

Theorem 5 *For every prime number $p > 3$, the set B_p has at least two components.*

Remark 6 We believe that B_3 is also an interval and Theorem 5 holds also for every integer $n > 3$. An interesting problem is how many components contained in B_p.

The Proof of Theorem 2

In order to prove Theorem 2, we need the following Lemmas. Set $h_r(x) := r^2 \exp(-2x) - x^2$ and $\Delta_r := \{z \in \mathbb{C} | \ |z| < r\}$.

Lemma 7 *Let $r \in (0, e^{-1})$, then h_r has 3 distinct zeros $x_1 < -1, x_2 \in (-1, 0)$ and $x_3 > 0$. Moreover, the solving set of inequality $h_r(x) \geq 0$ is the union of $I_1 = (-\infty, x_1]$ and $I_2 = [x_2, x_3]$.*

Proof Noting $f_0(x) = xe^x$ and $h_r(x) = e^{-2x}(r^2 - x^2e^{2x})$, we have

$$h_r(x) = 0 \Leftrightarrow |f_0(x)| = |r|, \tag{1}$$

and

$$h_r(x) > 0 \Leftrightarrow |f_0(x)| < |r|. \tag{2}$$

From $f_0'(x) = (x+1)e^x$, we see that $f_0(x)$ is decreasing in $(-\infty, -1]$ and increasing in $[-1, +\infty)$, and $f_0(-1) = -e^{-1}$ is the minimum value of $f_0(x)$. Note that

$$f_0(0) = 0, \quad \lim_{x \to -\infty} f_0(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} f_0(x) = +\infty,$$

if $r \in (0, e^{-1})$, then we infer that $f_0(x) = r$ has the only one root $x_3 > 0$, and $f_0(x) = -r$ has two roots $x_1 < -1$ and $x_2 \in (-1, 0)$. Moreover, the solving set of inequality $|f_0(x)| < |r|$ is the union of $(-\infty, x_1)$ and (x_2, x_3). Hence from (1) and (2), we obtain the assertion. □

Lemma 8 *Let $r \in (0, e^{-1})$, then $f_0^{-1}(\Delta_r)$ has two connected components D_1 and D_2, and the set $D_1 \cup D_2 \cup (-\infty, 0)$ is connected.*

Proof For every $z = x + iy \in f_0^{-1}(\Delta_r)$, we have

$$|f_0(z)| = |z \exp(z)| < r,$$

which implies

$$\sqrt{x^2 + y^2} \exp(x) < r.$$

It follows that

$$|y| < \sqrt{h_r(x)}.$$

From Lemma 7, we know that the graph of $|y| = \sqrt{h_r(x)}$ consists of two curves

$$L_1 : \ |y| = \sqrt{h_r(x)}, \quad x \in I_1$$

and

$$L_2 : \ |y| = \sqrt{h_r(x)}, \quad x \in I_2.$$

Therefore, $f_0^{-1}(\Delta_r)$ has two connected components D_1 and D_2, where $\partial D_1 = L_1$ and $\partial D_2 = L_2$. Obviously the set $D_1 \cup D_2 \cup (-\infty, 0)$ is connected. Hence we obtain the assertion. □

Lemma 9 *Let $I = (a, b)$ be an open interval, and $f : I \to I$ be a continuous mapping.*

(1) If $f(x) > x$ for every $x \in I$, then we have

$$\lim_{n \to +\infty} f^n(x) = b.$$

(2) If $f(x) < x$ for every $x \in I$, then we have

$$\lim_{n \to +\infty} f^n(x) = a.$$

Proof (1) Suppose $f(x) > x$ for every $x \in I$. Then it follows that the sequence $\{f^n(x)\}_{n=1}^{\infty}$ is increasing. Hence the sequence $\{f^n(x)\}_{n=1}^{\infty}$ either tends to $+\infty$ or tends to $x_0 < +\infty$. If the first case happens, then we have $b = +\infty$. If the second case happens, then we infer $x_0 = b$. Otherwise, $x_0 < b$, and then x_0 is a fixed point of f, which contradicts that $f(x) > x$ for every $x \in I$. Thus, we obtain that the sequence $\{f^n(x)\}_{n=1}^{\infty}$ tends to b.

(2) Similar as the proof of (1), we can obtain (2) easily. □

Proof of Theorem 2

Proof Let $\mu < 0$. Then singular value 0 of f_μ is an attracting fixed point. Denote the immediate attracting basin of 0 by D. For every $x < 0$, we have

$$0 > f_\mu(x) = x\exp(x+\mu) > x,$$

from Lemma 9, it follows that $\lim_{n\to+\infty} f_\mu^n(x) = 0$. Hence $x \in F(f_\mu)$, and then $(-\infty, 0) \subset D$.

Take r enough small such that $r_1 := re^{\mu} < -\mu$ and $r < e^{-1}$. Then $r_1 + \mu < 0$. For every $z \in \Delta_{r_1}$, we have

$$|f_\mu(z)| = |z\exp(z+\mu)| \le r_1\exp(r_1+\mu) < r_1.$$

This implies $\Delta_{r_1} \subset D$. Hence $f_\mu^{-1}(\Delta_{r_1}) \subset F(f_\mu)$.

It is easy to see that $|f_\mu(z)| < r_1 \Leftrightarrow |f_0(z)| < r$, which implies $f_\mu^{-1}(\Delta_{r_1}) = f_0^{-1}(\Delta_r)$. From Lemma 8, we know that $f_\mu^{-1}(\Delta_{r_1})$ has two connected components D_1 and D_2, the set $D_1 \cup D_2 \cup (-\infty, 0)$ is connected. Since $(-\infty, 0) \subset D$, we infer $f_\mu^{-1}(\Delta_{r_1}) \subset D$. Hence D is completely invariant. Since the Fatou set $F(f_\mu)$ has at most one cycle of periodic components, and has no wandering components, we have $F(f_\mu) = D$. Thus, Theorem 2 is proved completely. □

Dynamics of $f_\mu(x)$ for $\mu \le 2$ and the Proof of Theorem 3

For a real parameter μ, the attracting periodic points of f_μ (if they exist) are real. From now on, we suppose that the function f_μ only defined in $\mathbb{R}$.

It is known that f_μ has only two fixed points 0 and $-\mu$, the multiplier of 0 is e^{μ}, and the multiplier of $-\mu$ is $1-\mu$. We see that the periodic point 0 of f_μ is attracting (resp. parabolic) for $\mu < 0$ (resp. $\mu = 0$), the fixed point $-\mu$ of f_μ is attracting for $\mu \in (0, 2)$, and the fixed point $-\mu = -2$ of f_μ^2 is parabolic for $\mu = 2$. So the behavior of the iteration of f_μ for $\mu \le 2$ should be simple. Indeed, we have the following result.

Theorem 10 (1) *If* $\mu \le 0$, *then every point in* $(-\infty, -\mu)$ *is absorbed by the fixed point* 0 *and every point in* $(-\mu, +\infty)$ *escapes to* $+\infty$ *under iteration of* f_μ.

(2) *If* $\mu \in (0, 2]$, *then every point in* $(-\infty, 0)$ *is absorbed by the fixed point* $-\mu$ *and every point in* $(0, +\infty)$ *escapes to* $+\infty$ *under iteration of* f_μ.

Before proving Theorem 10, we first introduce some preliminary facts.

For the function f_μ, we have $f'_\mu(x) = (x+1)\exp(x+\mu)$, it follows that f_μ is decreasing in $(-\infty, -1]$ and increasing in $[-1, +\infty)$, and $s := f_\mu(-1)$ is the minimum value of f_μ. Moreover, we see that the following Claims hold.

Claim 1 *If* $\mu \leq 0$, *then*

(1) $f_\mu(x) > x$, *for every* $x \in (-\mu, +\infty)$;
(2) $0 < f_\mu(x) < x$, *for every* $x \in (0, -\mu)$, $\mu \neq 0$;
(3) $0 > f_\mu(x) > x$, *for every* $x \in (-\infty, 0)$.

Claim 2 *If* $\mu > 0$, *then*

(1) $f_\mu(x) > x$, *for every* $x \in (0, +\infty)$;
(2) $f_\mu(x) < x$, *for every* $x \in (-\mu, 0)$;
(3) $f_\mu(x) > x$, *for every* $x \in (-\infty, -\mu)$.

Since $f_\mu(x)$ is increasing in $[-1, +\infty)$ and $f_\mu(-\mu) = -\mu$, by Claim 2, we obtain the following Claim.

Claim 3 *If* $\mu \in (0, 1]$, *then*

(1) $x > f_\mu(x) > -\mu$, *for every* $x \in (-\mu, 0)$;
(2) $x < f_\mu(x) < -\mu$, *for every* $x \in (-1, -\mu)$, $\mu \neq 1$.

Let $g_\mu(x) := x + f_\mu(x) + 2\mu$. Then

$$f_\mu^2(x) = f_\mu(x)\exp(f_\mu(x) + \mu) = x\exp(x+\mu)\exp(f_\mu(x)+\mu) = x\exp(g_\mu(x)). \tag{3}$$

For the function $g_\mu(x)$, we have

$$g'_\mu(x) = 1 + (x+1)\exp(x+\mu) \tag{4}$$

and

$$g''_\mu(x) = (x+2)\exp(x+\mu). \tag{5}$$

From (5), we see that the curve $y = g_\mu(x)$ is convex in $(-\infty, -2]$ and concave in $[-2, +\infty)$. Furthermore, we have the following two Lemmas.

Lemma 11 *If* $\mu \leq 2$, *then the function* $g_\mu(x)$ *is increasing in* $(-\infty, +\infty)$.

Proof From (4), we have

$$g'_\mu(-2) = 1 - \exp(\mu - 2) \geq 0.$$

Since the curve $y = g_\mu(x)$ is convex in $(-\infty, -2]$ and concave in $[-2, +\infty)$, we infer that the function $g_\mu(x)$ is increasing in $(-\infty, +\infty)$. □

Lemma 12 *If* $\mu > 2$, *then the function* g_μ *has only three distinct zeros* $-\mu, p$ *and* q , *where*

$$p < -\mu \quad \text{and} \quad -2 < q < 0, \tag{6}$$

moreover

$$g'_\mu(p) > 0 \quad \text{and} \quad g'_\mu(q) > 0. \tag{7}$$

Proof From (4), we have

$$g'_\mu(-\mu) = 1 + (-\mu + 1)\exp(-\mu + \mu) = 2 - \mu < 0.$$

Note that $g_\mu(0) = 2\mu > 0$ and $g_\mu(-\mu) = 0$. Since the curve $y = g_\mu(x)$ is convex in $(-\infty, -2]$ and concave in $[-2, +\infty)$, we infer that the function g_μ has only three distinct zeros $-\mu, p$ and q , where $p < -\mu$ and $-2 < q < 0$, moreover, $g'_\mu(p) > 0$ and $g'_\mu(q) > 0$.

Proof of Theorem 10

Proof First, we prove the part (1) of Theorem 10.

From Claim 1, by Lemma 9, we infer that for $\mu \le 0$,

$$\lim_{n\to+\infty} f^n_\mu(x) = +\infty, \quad \text{for every } x \in (-\mu, +\infty),$$

and

$$\lim_{n\to+\infty} f^n_\mu(x) = 0, \quad \text{for every } x \in (-\infty, -\mu),$$

i.e., every point in $(-\mu, +\infty)$ escapes to $+\infty$ and every point in $(-\infty, -\mu)$ is absorbed by the fixed point 0 under iteration of f_μ. Thus, the part (1) of Theorem 10 is proved.

Next, we prove the part (2) of Theorem 10.

From Claim 2, by Lemma 9, we infer that every point in $(0, +\infty)$ escapes to $+\infty$ under iteration of f_μ for $\mu > 0$.

For every $x \in (-\infty, 0)$, we have $f_\mu(x) \in [s, 0)$. Hence, once we have proven that every point in $[s, 0)$ is absorbed by the fixed point $-\mu$ under iteration of f_μ, then it follows that every point in $(-\infty, 0)$ is also absorbed by the fixed point $-\mu$ under iteration of f_μ.

Note that $s > -1$ for $\mu \in (0, 1)$ and $s = -1$ for $\mu = 1$. Using Lemma 9, from Claim 3, we infer that for $\mu \in (0, 1]$, every point in $[s, 0)$ is absorbed by the fixed point $-\mu$ under iteration of f_μ, and then every point in $(-\infty, 0)$ is also absorbed by the fixed point $-\mu$ under iteration of f_μ.

The remainder to be proved is the following claim:

For $\mu \in (1, 2]$, every point in $[s, 0)$ is absorbed by the fixed point $-\mu$ under iteration of f_μ.

Suppose $\mu \in (1, 2]$. By Lemma 11, $g_\mu(x)$ is increasing in $(-\infty, +\infty)$, we have

$$g_\mu(x) > g_\mu(-\mu) = 0, \quad \text{for every } x \in (-\mu, 0).$$

Hence by (3), we have

$$f^2_\mu(x) = x\exp(g_\mu(x)) < x, \quad \text{for every } x \in (-\mu, 0). \tag{8}$$

Since $f_\mu(x)$ is decreasing in $(-\infty, -1]$, noting $f_\mu(-\mu) = -\mu$ and $f_\mu(s) = f_\mu^2(-1) < -1$, we have

$$s < f_\mu(x) < -\mu, \quad \text{for every } x \in (-\mu, -1) \tag{9}$$

and

$$-\mu < f_\mu(x) < -1, \quad \text{for every } x \in [s, -\mu). \tag{10}$$

From (8), (9) and (10), we obtain

$$-\mu < f_\mu^2(x) < x, \quad \text{for every } x \in (-\mu, -1).$$

Hence by Lemma 9, we get

$$\lim_{n\to+\infty} f_\mu^{2n}(x) = -\mu, \quad \text{for every } x \in (-\mu, -1),$$

and then

$$\lim_{n\to+\infty} f_\mu^{2n+1}(x) = f_\mu(-\mu) = -\mu, \quad \text{for every } x \in (-\mu, -1).$$

Thus we obtain

$$\lim_{n\to+\infty} f_\mu^{n}(x) = -\mu, \quad \text{for every } x \in [-\mu, -1).$$

Further, from (10), we have

$$\lim_{n\to+\infty} f_\mu^{n}(x) = -\mu, \quad \text{for every } x \in [s, -1). \tag{11}$$

i.e., every point in $[s, -1)$ is absorbed by the fixed point $-\mu$ under iteration of f_μ. For every $x \in [-1, 0)$, we have

$$f_\mu(x) = x\exp(x + \mu) < x.$$

Assume $f_\mu^n(x) \geq -1$ hold for all positive integer n. Then it follows that the sequence $\{f_\mu^n(x)\}_{n=1}^{\infty}$ is decreasing, hence it tends to a fixed point $x_0 \in [-1, 0)$ of f_μ. This contradicts that $f_\mu(x) = x\exp(x + \mu) < x$ for every $x \in [-1, 0)$. So there exists a positive integer k such that $f_\mu^k(x) \in [s, -1)$, and it follows from (11) that

$$\lim_{n\to+\infty} f^n(f_\mu^k(x)) = -\mu,$$

which implies that x is absorbed by the fixed point $-\mu$ under iteration of f_μ.

Thus, we completed the proof of Theorem 10. □

As a corollary of Theorem 10, we have the following result.

Theorem 13 *If $\mu \leq 2$, then f_μ has no periodic points of n -order for any $n \geq 2$.*

From (3) and Lemma 12, we immediately get the following result.

Lemma 14 *For every* $\mu > 2$, f_μ *has only one cycle of periodic points of 2-order.*

Let $\{p, q\}$ be the cycle of periodic points of 2-order of f_μ for $\mu > 2$, which satisfies (6) and (7). Note that $g_\mu(p) = 0$ and $q = f_\mu(p) = p\exp(p+\mu)$, which imply

$$p + q + 2\mu = 0, \tag{12}$$

then from (4), we have

$$g'_\mu(p) = 1 + (p+1)\exp(p+\mu) = 1 + \frac{q(p+1)}{p} = \frac{p+q+pq}{p}. \tag{13}$$

From (6), (7) and (13), we infer

$$p + q + pq < 0. \tag{14}$$

Let λ denote the multiplier of the cycle $\{p, q\}$. We have

$$\lambda = f'_\mu(p) \cdot f'_\mu(q) = (p+1)\exp(p+\mu) \cdot (q+1)\exp(q+\mu).$$

It follows from (12) that

$$\lambda = (p+1)(q+1) = p + q + pq + 1. \tag{15}$$

Hence by (14), we get $\lambda < 1$. Furthermore, we have the following result.

Lemma 15 *The multiplier* λ *as a function of* μ *defined in* $(2, +\infty)$ *is decreasing, and its range is* $(-\infty, 1)$.

Proof From (12), we have

$$\frac{dp}{d\mu} + \frac{dq}{d\mu} + 2 = 0. \tag{16}$$

Moreover, since $q = f_\mu(p) = p\exp(p+\mu)$, i.e., $\log(q/p) = p + \mu$, we have

$$\frac{1}{q} \cdot \frac{dq}{d\mu} - \frac{1}{p} \cdot \frac{dp}{d\mu} = \frac{dp}{d\mu} + 1. \tag{17}$$

From (16) and (17), by direct calculation, we obtain

$$\frac{dp}{d\mu} = -\frac{p(q+2)}{p+q+pq} \tag{18}$$

and

$$\frac{dq}{d\mu} = -\frac{q(p+2)}{p+q+pq}. \tag{19}$$

From (15), we have

$$\frac{d\lambda}{d\mu} = \frac{dp}{d\mu} + \frac{dq}{d\mu} + q\frac{dp}{d\mu} + p\frac{dq}{d\mu}.$$

Then by (16), (18) and (19), we get

$$\frac{d\lambda}{d\mu} = -2 - \frac{pq(p+q+4)}{p+q+pq}. \tag{20}$$

By (12) and $\mu > 2$, we have $p+q+4 = -2\mu+4 < 2$. Hence by (6), (14) and (20), we infer $\frac{d\lambda}{d\mu} < 0$, thus λ as a function of μ is decreasing.

Following (6), (14) and (18), we get $\frac{dp}{d\mu} < 0$, hence p as a function of μ is decreasing, and then $\lim_{\mu\to 2^+} p$ exists, say p_0. Then from $g_\mu(p) = 0$, we have $g_2(p_0) = 0$. Since the function $g_2(x)$ is increasing in $(-\infty, +\infty)$, we get $p_0 = -2$. Hence from (12) and (15), we have

$$\lim_{\mu\to 2^+} q = \lim_{\mu\to 2^+} (-p-2\mu) = -2$$

and

$$\lim_{\mu\to 2^+} \lambda = \lim_{\mu\to 2^+} (p+1)(q+1) = 1. \tag{21}$$

By (6) and (12), we have

$$|q| = |p\exp(p+\mu)| = \frac{|q+2\mu|}{\exp(q+\mu)} < \frac{2+2\mu}{\exp(\mu-2)}.$$

It is easy to obtain by calculating that

$$\lim_{\mu\to+\infty} \frac{2+2\mu}{\exp(\mu-2)} = 0,$$

which implies $\lim_{\mu\to+\infty} q = 0$. Hence from (12) and (15), we have

$$\lim_{\mu\to+\infty} \lambda = \lim_{\mu\to+\infty} (-q-2\mu+1)(q+1) = -\infty. \tag{22}$$

Since λ is decreasing, by (21) and (22), we obtain that the range of λ is $(-\infty, 1)$. □

Proof of Theorem 3

Proof Set $\mu_* := \lambda^{-1}(-1)$. Then by Lemma 15, we have $\mu_* \neq +\infty$ and $\lambda^{-1}(-1,1) = (2, \mu_*)$. By Theorem 13 and Lemma 14, we deduce that $\mu \in B_2$ if and only if $\mu > 2$ and $\lambda(\mu) \in (-1, 1)$. Hence we obtain $B_2 = (2, \mu_*)$.

Therefore, Theorem 3 is proved completely. □

Remark 16 Computation of μ_*.

From (12) and (15), we have

$$\lambda = (1-2\mu-q)(1+q) \quad \Leftrightarrow \quad (q+\mu)^2 = (\mu-1)^2 - \lambda.$$

Noting $q \in (-2, 0)$ and $\mu > 2$, we obtain

$$q = \sqrt{(\mu-1)^2 - \lambda} - \mu,$$

and then

$$p = -\sqrt{(\mu-1)^2-\lambda} - \mu.$$

Hence the equation $p = q\exp(q+\mu)$ with $\lambda(\mu_*) = -1$ implies that μ_* is the root of the function

$$\Phi(\mu) = \frac{w+\mu}{w-\mu} + \exp(w) \quad \text{with} \quad w := \sqrt{(\mu-1)^2+1}.$$

The plot of $\Phi(\mu)$ is as in Fig. 1. One can compute the root of $\Phi(\mu)$ up to machine precision with numerical methods like bisection, secant method and so on.

Result: $\mu_* = 2.526467725\cdots$
(with $p = -4.351324903\cdots$ and $q = -0.701610548\cdots$).

Remark 17 The Taylor series expansion of $\lambda(\mu)$.

From (12), (15) and (20), we have

$$\begin{aligned}\frac{d\lambda}{d\mu} &= -2 - \frac{pq(p+q+4)}{p+q+pq}\\ &= -2 - \frac{(2\mu+\lambda-1)(4-2\mu)}{\lambda-1}\\ &= \frac{2(\lambda-1)((\mu-2)-1)+4(\mu-2)^2+8(\mu-2)}{\lambda-1},\end{aligned}$$

which implies that

$$(\lambda-1)\frac{d\lambda}{d\mu} = 2(\lambda-1)((\mu-2)-1)+4(\mu-2)^2+8(\mu-2). \tag{23}$$

Suppose that the formal Taylor series expansion with expansion point 2 is

$$\lambda(\mu) = \sum_{k=0}^{\infty} a_k(\mu-2)^k,$$

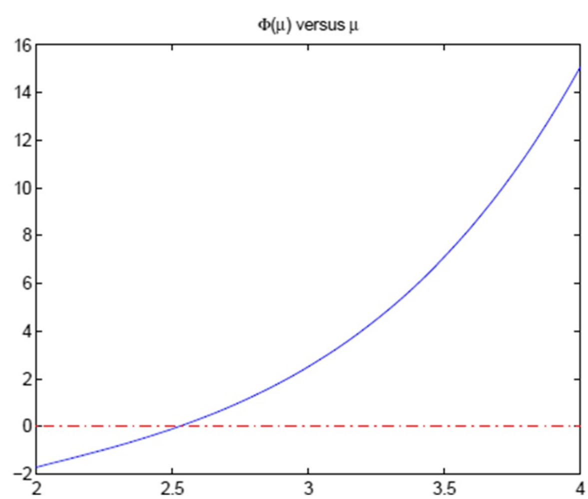

Fig. 1 $\Phi(\mu)$, $2 \le \mu \le 4$

then

$$\frac{d\lambda(\mu)}{d\mu}=\sum_{k=1}^{\infty}ka_k(\mu-2)^{k-1}.$$

Note that $a_0=\lambda(2^+)=1$, and let $t:=(\mu-2)$, from (23) we have

$$\left(\sum_{k=1}^{\infty}a_kt^k\right)\left(\sum_{k=1}^{\infty}ka_kt^{k-1}\right)=2\sum_{k=1}^{\infty}a_kt^{k+1}-2\sum_{k=1}^{\infty}a_kt^k+4t^2+8t,$$

which implies that

$$\sum_{l=1}^{\infty}\left(\sum_{j=1}^{l}ja_ja_{l+1-j}\right)t^l=t(8-2a_1)+t^2(2a_1-2a_2+4)+\sum_{l=3}^{\infty}2(a_{l-1}-a_l)t^l.$$

The comparison of the left and the right side with respect to t^l yields

$$\begin{aligned}
&l=1:\quad a_1^2=8-2a_1\ \Rightarrow a_1=-1\pm 3\ \Rightarrow \frac{d\lambda(\mu)}{d\mu)}|_{\mu=2}=a_1=-4<0;\\
&l=2:\quad a_1a_2+2a_2a_1=2a_1-2a_2+4\ \Rightarrow a_2=0.4;\\
&l\geq 3:\quad a_l=\frac{1}{4l+2}\left(\sum_{j=2}^{l-1}ja_ja_{l+1-j}-2a_{l-1}\right).
\end{aligned}$$

The Plot of the Taylor series expansion of $\lambda(\mu)$ up to 10th order with expansion point 2 is as in Fig. 2. Indeed, the Taylor polynomials can also be used to approximate μ_*.

The Proof of Theorem 4

In this section we prove Theorem 4 by finding a parameter μ_n such that f_{μ_n} has super-attracting periodic points of n-order. This parameter μ_n satisfies the equation $f_\mu^n(-1)=-1$, i.e., μ belongs to the set

$$E_n:=\{\mu\in\mathbb{R}|s_n(\mu)=-1\},$$

where $s_n(\mu):=f_\mu^n(-1)$ for every positive integer n.

Lemma 18 (Jang (1992)) *Let $n\geq 2$, then $s_n(\mu)\to 0$ as $\mu\to+\infty$.*

Lemma 19 *For every positive integer n , E_n is a finite set, and $E_1=\{1\}\subset E_n$.*

Proof Since $s_1(\mu)=f_\mu(-1)=-\exp(\mu-1)$, we get $E_1=\{1\}$, and from $s_n(1)=-1$, we get $\{1\}\subset E_n$.

Now, suppose $n\geq 2$. Since $s_1(\mu)=f_\mu(-1)$ is the minimum value of f_μ, we have

$$s_n(\mu)\geq f_\mu(-1)=-\exp(\mu-1)>-1,\quad \text{for every}\quad \mu<1.$$

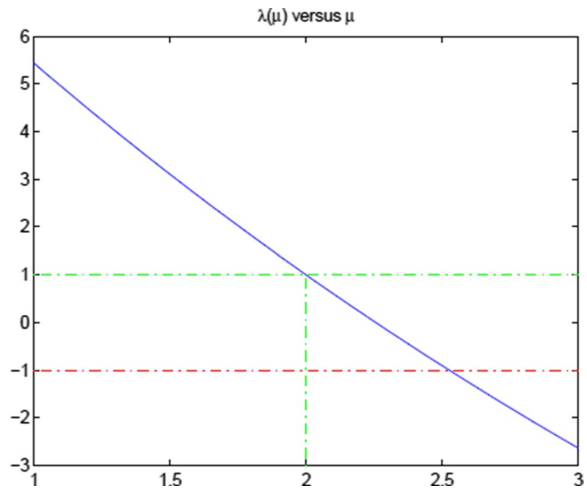

Fig. 2 Taylor series expansion of $\lambda(\mu)$

Hence $\mu \geq 1$ for every $\mu \in E_n$. From Lemma 18, there is M_n such that $s_n(\mu) > -1$ for every $\mu > M_n$. Hence $\mu \leq M_n$ for every $\mu \in E_n$. Thus E_n is a bounded set, and then is a finite set. □

Lemma 19 allows us to define $\mu_n := \max\{\mu \in E_n\}$. Clearly, $\mu_1 = 1$ and $\mu_n \geq 1$ for every $n \geq 2$.

Lemma 20 *Let $n \geq 2$. If $\mu > \mu_n$, then $s_n(\mu) > -1$ and $s_{n+1}(\mu) < s_n(\mu)$.*

Proof It is easy to see that $s_n(\mu) < 0$ for every μ. Set

$$I_n := \{s_n(\mu) | \mu \in (\mu_n, +\infty)\}.$$

Clearly, I_n is an interval, and $I_n \subset (-\infty, 0)$. Note that $-1 \notin I_n$ from the definition of μ_n. Hence by Lemma 18 and $s_n(\mu_n) = -1$, we infer $I_n = (-1, 0)$. Hence if $\mu > \mu_n$, then $-1 < s_n(\mu) < 0$. Noting $\mu_n \geq 1$, we have

$$s_{n+1}(\mu) = s_n(\mu) \exp(s_n(\mu) + \mu) < s_n(\mu) \exp(\mu_n - 1) \leq s_n(\mu).$$

Thus we obtain the assertion. □

The following two Lemmas have been proved by Kuroda and Jang (1997). Here we give different proofs of them.

Lemma 21 *The sequence $\{\mu_n\}_{n=1}^{\infty}$ is increasing.*

Proof Since $\mu_n \geq 1$ and $s_n(\mu_n) = -1$ for every $n \geq 1$, we have

$$s_{n+1}(\mu_n) = s_n(\mu_n) \exp(s_n(\mu_n) + \mu_n) = -\exp(\mu_n - 1) \leq -1.$$

Since

$$\frac{ds_{n+1}}{d\mu} = ((1 + s_n) \frac{ds_n}{d\mu} + s_n) \exp(s_n + \mu),$$

we get

$$\frac{ds_{n+1}}{d\mu}|_{\mu=\mu_n} = -\exp(\mu_n - 1) < 0.$$

Hence there exists $\mu_n' > \mu_n$ such that $s_{n+1}(\mu_n') < s_{n+1}(\mu_n) \le -1$. Following Lemma 20, we deduce $\mu_n' < \mu_{n+1}$, and then $\mu_n < \mu_{n+1}$. Thus $\{\mu_n\}_{n=1}^{\infty}$ is increasing. □

Lemma 22 *For every positive integer n , the function f_{μ_n} has super-attracting periodic points of n-order.*

Proof Clearly, f_{μ_1} has the super-attracting fixed point -1. Since $\mu_2 > 1$, we have

$$s_1(\mu_2) = -\exp(\mu_2 - 1) < -1 = s_2(\mu_2).$$

This implies that $s_1(\mu_2), s_2(\mu_2)$ are super-attracting periodic points of 2-order of f_{μ_2}.

Let $n \ge 3$ and $k = 2, 3, \ldots, n-1$. Then $\mu_n > \mu_k > 1$ from Lemma 21. Hence by Lemma 20, we have

$$s_1(\mu_n) = -\exp(\mu_n - 1) < -1 = s_n(\mu_n) < s_{n-1}(\mu_n) < \cdots < s_2(\mu_n).$$

This implies that $s_1(\mu_n), s_2(\mu_n), \cdots, s_n(\mu_n)$ are super-attracting periodic points of n-order of f_{μ_n}. □

From Lemma 22, we immediately get the following Theorem.

Theorem 23 *For every positive integer n , $\mu_n \in B_n$.*

Thus, Theorem 4 follows from Theorem 23.

The Proof of Theorem 5

Let $\hat{B}_n$ be the set of real parameters μ such that f_μ has a periodic point of n-order, whose multiplier is less than 1. It follows from the implicit function theorem that the set $\hat{B}_n$ is an open set for every positive integer n.

Theorem 24 *Suppose that p is a prime number, (a, b) is a component of $\hat{B}_p$. Then f_a has parabolic periodic points of p-order. Furthermore, $a \in \partial B_p$.*

Proof From Theorem 13, we know that if $\mu \le 2$, then f_μ has no periodic points of p-order. Hence we have $a \ge 2$. Choose a decreasing sequence $\{a_n\}_{n=1}^{\infty} \subset (a, b)$ such that $\lim_{n\to\infty} a_n = a$, and choose a periodic point x_n of p-order of f_{a_n} with multiplier $\lambda_n < 1$. From Claim 2 in "Dynamics of $f_\mu(x)$ for $\mu \le 2$ and the Proof of Theorem 3" section, we have $x_n < 0$. Since $s_1(\mu)$ is the minimum value of f_μ and $s_1(\mu) = -\exp(\mu - 1)$ as a function of μ is decreasing, we have

$$x_n \in [s_1(a_n), 0] \subset [s_1(a_1), 0].$$

Thus we may suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, otherwise we consider a subsequence of $\{x_n\}_{n=1}^{\infty}$. Set $x_0 := \lim_{n\to\infty} x_n$, then we have

$$f_a^p(x_0) = \lim_{n\to\infty} f_{a_n}^p(x_n) = \lim_{n\to\infty} x_n = x_0$$

and

$$\lambda_0 := \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} (f_{a_n}^p)'(x_n) = (f_a^p)'(x_0).$$

Hence x_0 is a fixed point of f_a^p, and its multiplier $\lambda_0 \leq 1$ from $\lambda_n < 1$. Since p is a prime number, x_0 is either fixed point or periodic point of p-order of f_a.

Assume x_0 is not a periodic point of p-order of f_a, then x_0 is a fixed point of f_a. Thus $x_0 = 0$ or $x_0 = -a$. Noting $(f_a^p)'(0) = e^{pa} > 1$, we have $x_0 = -a$. Define the function

$$F(x, \mu) := f_\mu^p(x) - x, \quad (x, \mu) \in \mathbb{R}^2.$$

We have

$$F_x'(x_0, a) = (f_a^p)'(x_0) - 1 = (1 - a)^p - 1 < 0.$$

Hence there exists a disc D which is a neighborhood of (x_0, a) such that $F_x'(x, \mu) < 0$ for every $(x, \mu) \in D$. Choose a positive integer m such that $(x_m, a_m) \in D$ and $(-a_m, a_m) \in D$. Note that $x_m \neq -a_m$, and $F(x_m, a_m) = 0, F(-a_m, a_m) = 0$. Then by Rolle theorem, there exists a point $(y_0, a_m) \in D$ such that $F_x'(y_0, a_m) = 0$. It contradicts that $F_x'(y_0, a_m) < 0$. Hence x_0 is a periodic point of p-order of f_a.

Because of $a \notin \hat{B}_p$, we infer that $\lambda_0 = 1$. Hence x_0 is a parabolic periodic point of p-order of f_a. Since $\lambda_0 = \lim_{n\to\infty} \lambda_n$, we infer that $a_n \in B_p$ for large enough n. This implies $a \in \partial B_p$. Hence, Theorem 24 is completed. □

The proof of Theorem 5 needs Theorem 24 and the following Lemmas.

Lemma 25 (Li and York 1975, Li-York theorem) *Let I be a closed interval, and $f : I \to I$ be a continuous mapping. If f has periodic points of 3-order, then f has periodic points of n -order for every positive integer n.*

Lemma 26 (Deng and Cai (1993)) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Suppose that f has two fixed points x_1, x_2, say $x_1 < x_2$. If their multipliers are greater than 1, then f has another fixed point $x_3 \in (x_1, x_2)$.*

Proof of Theorem 5

Proof By assumption, the prime number $p \geq 5$.

Set $I_0 := [s_1(\mu_3), 0]$ and $g := f_{\mu_3}|I_0$. Since $s_1(\mu_3)$ is the minimum value of f_{μ_3}, we see that g is a self-mapping of I_0. From Lemma 22, g has super-attracting periodic points of 3-order. According to Lemma 25, g has periodic points of p-order. Since the Fatou set $F(f_\mu)$ has at most one cycle of periodic components, we get that all of the periodic points of p-order of g are repelling. Let n_p denote the number of the cycles of periodic points of p-order of g , and n_p' (resp. n_p'') denote the number of the cycles of which the multipliers are greater (resp. less) than 1. Clearly, $1 \leq n_p < +\infty$. Since the periodic points of p-order of g are repelling, we have $n_p' + n_p'' = n_p$. Since p is a prime number, every fixed point of g^p is either fixed point or periodic point of p-order of g. Noting that g has only two fixed points 0 and $-\mu$, we get that g_μ^p has $pn_p + 2$ fixed points. Assume $n_p'' = 0$, then g_μ^p has at least pn_p fixed points, whose multipliers are greater than 1. However, by Lemma 26, we deduce that g_μ^p has at least $2pn_p - 1$ fixed points, and then $pn_p + 2 \geq 2pn_p - 1$, which contradicts that $pn_p \geq p \geq 5$. Hence we obtain $n_p'' \neq 0$, which implies $\mu_3 \in \hat{B}_p$.

Let (a, b) be the component of $\hat{B}_p$, which contains μ_3. Then by Theorem 24, there exists a_0 such that $a_0 < \mu_3$ and $a_0 \in B_p$. From Lemma 21, $\mu_3 < \mu_p$. Since $\mu_3 \notin B_p$, we infer that there exist two different components of B_p, one of them contains a_0, the other contains μ_p.

Thus, Theorem 5 is proved completely. □

Conclusions

It is known that the dynamics given by the iteration of transcendental entire maps has been widely studied. In this paper, we consider the dynamics of the functions $f_\mu(z) = ze^{z+\mu}$ with the real parameter, and prove that the Fatou set $F(f_\mu)$ is a completely invariant attracting basin for every parameter $\mu < 0$. We say that a real parameter μ belongs to the set B_n for a positive integer n if f_μ has an attracting cycle of n-order. Regarding the set B_n for $n > 1$, we show that (1) there exists $\mu_* \neq +\infty$ such that $B_2 = (2, \mu_*)$; (2) for every positive integer $n > 2$, the set B_n is non-empty; (3) for every prime number $p > 3$, the set B_p has at least two components.

Authors' contributions

The main idea of this paper was proposed by XCD, FNM, JML and WJY, prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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