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Hermite–Hadamard type inequalities for n -times differentiable and geometrically quasi-convex functions

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Abstract

By Hölder's integral inequality, the authors establish some Hermite–Hadamard type integral inequalities for n -times differentiable and geometrically quasi-convex functions.

Keywords: Hermite–Hadamard type inequality, geometrically quasi-convex function, Hölder integral inequality

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Background

Let I be an interval on $\mathbb{R} = (-\infty, \infty)$. A function $f : I \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1)$$

for $x, y \in I$ and $\lambda \in [0, 1]$. If the inequality (1) reverses, then f is said to be concave on I .

A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}_+$ is said to be geometrically convex on I if

$$f(x^\lambda y^{1-\lambda}) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}$$

for $x, y \in I$ and $\lambda \in [0, 1]$.

One of the most famous inequalities for convex functions is Hermite–Hadamard's inequality: if $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is convex on an interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}; \quad (2)$$

if f is concave on I , then the inequality (2) is reversed.

We now collect several Hermite–Hadamard type integral inequalities as follows.

Theorem 1 (Dragomir and Agarwal 1998) *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)[|f'(a)| + |f'(b)|]}{8}.$$

Theorem 2 (Xi and Qi 2013) *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is geometrically convex on $[a, b]$, then*

$$\left| \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx - f(\sqrt{ab}) \right| \leq \frac{\ln b - \ln a}{4} \left\{ L\left([a|f'(a)|]^{1/2}, [b|f'(b)|]^{1/2}\right) \right\}^2,$$

where

$$L(u, v) = \frac{u - v}{\ln u - \ln v}$$

for $u, v > 0$ and $u \neq v$ is called the logarithmic mean.

Theorem 3 (Dragomir and Agarwal 1998) *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ for $q \geq 1$ is convex on $[a, b]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}.$$

Theorem 4 (Kirmaci 2004) *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ for $p > 1$ is convex on $[a, b]$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} \left\{ \left[|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}\right]^{1-1/p} + \left[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\right]^{1-1/p} \right\}.$$

Corresponding to the concept of geometrically convex functions, the geometrically quasi-convex functions were introduced in Qi and Xi (2014) as follows.

Definition 1 (Definition 2.1 Qi and Xi 2014) *A function $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_0 = [0, \infty)$ is said to be geometrically quasi-convex function on I if*

$$f(x^\lambda y^{1-\lambda}) \leq \sup\{f(x), f(y)\}$$

for $x, y \in I$ and $\lambda \in [0, 1]$.

In Qi and Xi (2014), some integral inequalities of Hermite–Hadamard type for geometrically quasi-convex functions were established.

In recent years, some other kinds of Hermite–Hadamard type inequalities were generated. For more systematic information, please refer to Bai et al. (2012), Pearce and Pečarić (2000), Pečarić and Tong (1991), Wang and Qi (2013), Wang et al. (2012), Xi et al. (2012) and related references therein.

The aim of this paper is to find more integral inequalities of Hermite–Hadamard type for n -times differentiable and geometrically quasi-convex functions.

A Lemma

In order to obtain our main results, we need the following Lemma.

Lemma 1 (Wang and Shi 2016) For $n \in \mathbb{N}$, let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a n -times differentiable function on I° and $a, b \in I$ with $a < b$. If $f^{(n)} \in L_1([a, b])$, then

$$\begin{aligned} & \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \\ &= \frac{(-1)^{n-1} (\ln b - \ln a)}{n!} \int_0^1 a^{(n+1)t} b^{(n+1)(1-t)} f^{(n)}(a^t b^{1-t}) dt. \end{aligned}$$

Remark 1 Under the conditions of Lemma 1, taking $n = 1$, we obtain

$$bf(b) - af(a) - \int_a^b f(x) dx = (\ln b - \ln a) \int_0^1 a^{2t} b^{2(1-t)} f'(a^t b^{1-t}) dt,$$

which can be found in Zhang et al. (2013).

Inequalities for geometrically quasi-convex functions

Now we start out to establish some new Hermite–Hadamard type inequalities for n -times differentiable and geometrically quasi-convex functions.

Theorem 5 For $n \in \mathbb{N}$, suppose that $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ is a n -times differentiable function on I° , that $f^{(n)} \in L_1([a, b])$, and that $a, b \in I$ with $a < b$. If $|f^{(n)}|^q$ is geometrically quasi-convex on $[a, b]$ for $q \geq 1$, then

$$\begin{aligned} & \left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \\ & \leq \frac{(\ln b - \ln a)}{n!} L(a^{n+1}, b^{n+1}) \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\}. \end{aligned}$$

Proof By the geometric quasi-convexity of $|f^{(n)}|^q$, Lemma 1, and Hölder’s inequality, one has

$$\begin{aligned} & \left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \\ & \leq \frac{\ln b - \ln a}{n!} \int_0^1 a^{(n+1)t} b^{(n+1)(1-t)} |f^{(n)}(a^t b^{1-t})| dt \\ & \leq \frac{\ln b - \ln a}{n!} \left[\int_0^1 a^{(n+1)t} b^{(n+1)(1-t)} dt \right]^{1-1/q} \\ & \quad \times \left\{ \int_0^1 a^{(n+1)t} b^{(n+1)(1-t)} \sup\{|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\} dt \right\}^{1/q} \\ & = \frac{(\ln b - \ln a) L(a^{n+1}, b^{n+1})}{n!} \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\}. \end{aligned}$$

Theorem 5 is thus proved. □

Corollary 1 Under the assumptions of Theorem 5, if $q = 1$, then

$$\left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \leq \frac{\ln b - \ln a}{n!} L(a^{(n+1)}, b^{(n+1)}) \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\}.$$

Theorem 6 For $n \in \mathbb{N}$, suppose that $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ is a n -times differentiable function on I° , that $f^{(n)} \in L_1([a, b])$, and that $a, b \in I$ with $a < b$. If $|f^{(n)}|^q$ is geometrically quasi-convex on $[a, b]$ for $q > 1$, then

$$\left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \leq \frac{\ln b - \ln a}{n!} \times \left[L\left(a^{\frac{q(n+1)-m}{q-1}}, b^{\frac{q(n+1)-r}{q-1}}\right) \right]^{1-1/q} [L(a^m, b^r)]^{1/q} \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\}$$

for $0 \leq m, r \leq (n + 1)q$.

Proof From the geometric quasi-convexity of $|f^{(n)}|^q$, Lemma 1, and Hölder’s inequality, we have

$$\begin{aligned} & \left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \\ & \leq \frac{\ln b - \ln a}{n!} \int_0^1 a^{(n+1)t} b^{(n+1)(1-t)} |f^{(n)}(a^t b^{1-t})| dt \\ & \leq \frac{\ln b - \ln a}{n!} \left[\int_0^1 a^{[q(n+1)-m]t/(q-1)} b^{[q(n+1)-r](1-t)/(q-1)} dt \right]^{1-1/q} \\ & \quad \times \left\{ \int_0^1 a^{mt} b^{r(1-t)} \sup\{|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\} dt \right\}^{1/q} \\ & = \frac{\ln b - \ln a}{n!} \left[L\left(a^{\frac{q(n+1)-m}{q-1}}, b^{\frac{q(n+1)-r}{q-1}}\right) \right]^{1-1/q} [L(a^m, b^r)]^{1/q} \\ & \quad \times \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\}. \end{aligned}$$

The proof of Theorem 6 is complete. □

Corollary 2 Under the conditions in Theorem 6,

1. if $m = r = 0$, then

$$\left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \leq \frac{\ln b - \ln a}{n!} \left[L\left(a^{\frac{q(n+1)}{q-1}}, b^{\frac{q(n+1)}{q-1}}\right) \right]^{1-1/q} \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\};$$

2. if $m = r = q(n + 1)$, then

$$\left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \leq \frac{\ln b - \ln a}{n!} [L(a^{q(n+1)}, b^{q(n+1)})]^{1/q} \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\};$$

3. if $m = 0$ and $r = q(n + 1)$, then

$$\left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \leq \frac{\ln b - \ln a}{n!} \times [L(a^{\frac{q(n+1)}{q-1}}, 1)]^{1-1/q} [L(1, b^{q(n+1)})]^{1/q} \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\};$$

4. if $m = n + 1$ and $r = q(n + 1)$, then

$$\left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \leq \frac{\ln b - \ln a}{n!} \times [L(a^{n+1}, 1)]^{1-1/q} [L(a^{n+1}, b^{q(n+1)})]^{1/q} \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\};$$

5. if $m = q(n + 1)$ and $r = 0$, then

$$\left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \leq \frac{\ln b - \ln a}{n!} \times [L(1, b^{\frac{q(n+1)}{q-1}})]^{1-1/q} [L(a^{q(n+1)}, 1)]^{1/q} \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\};$$

6. if $m = q(n + 1)$ and $r = n + 1$, then

$$\left| \sum_{k=1}^n \frac{(-1)^{k-1}}{k!} [b^k f^{(k-1)}(b) - a^k f^{(k-1)}(a)] - \int_a^b f(x) dx \right| \leq \frac{\ln b - \ln a}{n!} \times [L(1, b^{n+1})]^{1-1/q} [L(a^{q(n+1)}, b^{n+1})]^{1/q} \sup\{|f^{(n)}(a)|, |f^{(n)}(b)|\}.$$

Conclusion

Our main results in this paper are those integral inequalities of Hermite–Hadamard type in Theorems 5 and 6 and Corollaries 1 and 2.

Authors’ contributions

JZ, FQ, G-CX and Z-LP contributed equally to the manuscript. All authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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